163. Find all positive integers m and n such that the integer

$$\underbrace{2\dots2}_{m\text{ times}}\underbrace{5\dots5}_{n\text{ times}}$$

is a perfect square.

Solution. Since

$$\underbrace{2\dots2}_{m\text{ times}}\underbrace{5\dots5}_{n\text{ times}} = 2\left(\frac{10^m-1}{9}\right)10^n + 5\left(\frac{10^n-1}{9}\right),$$

the problem reduces to determining all $(m, n, x) \in \mathbb{N}^3$ such that

$$2\left(\frac{10^m-1}{9}\right)10^n + 5\left(\frac{10^n-1}{9}\right) = x^2.$$

Clearing the denominators in the left-hand-side of this equation and grouping the like terms afterwards, the equation under consideration becomes

$$(1) \qquad 2\cdot 10^{m+n} + 3\cdot 10^n - 5 = 9x^2.$$

In the light of the fact that for any $m, n \in \mathbb{N}$, with $n \geq 3$, the expression in the left-hand-side of (1) is congruent to -5 modulo 8 whereas the right-hand-side is either $0, 1$ or 4 modulo 8, we see that, if $(m, n, x) \in \mathbb{N}^3$ is a solution of (1), then $n = 2$ or $n = 1$.

The case $n = 2$ can be discarded by an analogous analysis modulo 8, too: in this situation, the left-hand-side of (1) is congruent to 7 modulo 8.

In the case $n = 1$, equation (1) becomes

$$(2) \qquad 5^2(2^{m+2}\cdot 5^{m-1} + 1) = 9x^2.$$

When $m = 1$ we obtain the solution $(m = 1, n = 1, x = 5)$. In the event that $m > 1$, we proceed as follows. Firstly, we notice that equation (2) can be rewritten as

$$2^{m+2}\cdot 5^{m-1} = (3X - 1)(3X + 1)$$

for some odd natural number X; from this and the observation that $(3X - 1, 3X + 1) = 2$, we see that we only need to consider the following two subcases:

a) $3X - 1 = 2^\alpha \cdot 5^{m-1}$, $3X + 1 = 2^\beta$ for some $\alpha, \beta \in \mathbb{N}$ such that $\alpha + \beta = m + 2$.

b) $3X - 1 = 2^\alpha$, $3X + 1 = 5^{m-1}\cdot 2^\beta$ for some $\alpha, \beta \in \mathbb{N}$ such that $\alpha + \beta = m + 2$.

The equations in **a** imply that $2 + 2^\alpha \cdot 5^{m-1} = 2^\beta$; from this and the straightforward inequalities $2^\alpha < 2 + 2^\alpha\cdot 5^{m-1} = 2^\beta$, we infer that $\alpha = 1$ (otherwise, $2 + 2^\alpha \cdot 5^{m-1} \equiv 2 \pmod 4$ while $2^\beta \equiv 0 \pmod 4$). Hence, m satisfies $2 = 2^{m+1} - 2\cdot 5^{m-1}$, wherefrom we obtain the inequality $(5/2)^m < 5$, which is absurd since we are pondering the case in which $m > 1$.

On the other hand, the conditions in **b** give that $2 + 2^\alpha = 5^{m-1}\cdot 2^\beta$. This equality allows us to infer that, in this subcase, β cannot be greater than 1. Hence, we arrive at the equation $2 + 2^{m+1} = 5^{m-1}\cdot 2$, from which we distill the inequality $1 + 2^m = 5^{m-1} > 2^{2(m-1)}$. Since in the range $m > 1$, the latter inequality holds true only when $m = 2$, we have a second solution to the equation in (1): $(m = 2, n = 1, x = 15)$.

In conclusion, there are only two natural numbers of the form in question: $25 = 5^2$ and $225 = 15^2$.

□

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