171.[*] Prove that every integer can be written in infinitely many ways in the form

$$\pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2k+1)^2$$

for some choice of the signs + and −.

Solution. It suffices to prove the assertion for non-negative n's because if $n \in \mathbb{N}$ and $n = \pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2k+1)^2$ for some $k \in \mathbb{Z}^+$ and a suitable choice of the signs + and −, then $-n$ is a nonpositive integer and $-n = \mp 1^2 \mp 3^2 \mp 5^2 \mp \cdots \mp (2k+1)^2$.

Firstly, we show that every $n \in \mathbb{Z}^+$ can be written in at least one way in the form under consideration. Since

$$16 = (u-5)^2 - (u-3)^2 - (u-1)^2 + (u+1)^2 \tag{1}$$

for every $u \in \mathbb{Z}$, it follows that if $n \in \mathbb{Z}^+$ and

$$n = \pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2k+1)^2 \tag{2}$$

for some $k \in \mathbb{Z}$ and a suitable choice of the signs + and −, then

$$n + 16 = \pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2k+1)^2 + \underbrace{(2k+3)^2 - (2k+5)^2 - (2k+7)^2 + (2k+9)^2}_{=16}.$$

Hence, in order to conclude that every non-negative integer n can be written in at least one way in the desired form, we only need to verify that every $n \in \{0, 1, \ldots, 15\}$ can be so represented. Aided by Mathematica®, we find that

$$\begin{aligned}
0 &= -1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 \\
1 &= 1^2 \\
2 &= 1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 \\
3 &= 1^2 + 3^2 + 5^2 + 7^2 - 9^2 \\
4 &= -1^2 - 3^2 - 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 - 17^2 + 19^2 \\
5 &= 1^2 + 3^2 + 5^2 + 7^2 - 9^2 + 11^2 + 13^2 + 15^2 + 17^2 - 19^2 - 21^2 \\
6 &= -1^2 - 3^2 + 5^2 - 7^2 - 9^2 + 11^2 \\
7 &= 1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 + 17^2 - 19^2 + 21^2 - 23^2 - 25^2 - 27^2 + 29^2 \\
8 &= -1^2 + 3^2 \\
9 &= -1^2 - 3^2 - 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 - 17^2 + 19^2 - 21^2 + 23^2 + 25^2 + 27^2 - 29^2 \\
&+ 31^2 - 33^2 - 35^2 + 37^2 \\
10 &= 1^2 + 3^2 \\
11 &= -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 + 17^2 - 19^2 - 21^2 + 23^2 + 25^2 \\
12 &= -1^2 - 3^2 - 5^2 - 7^2 - 9^2 + 11^2 - 13^2 + 15^2 \\
13 &= -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 - 15^2 + 17^2 \\
14 &= -1^2 - 3^2 - 5^2 + 7^2 \\
15 &= -1^2 - 3^2 + 5^2;
\end{aligned}$$

which is just what we wanted.

Finally, from (1) it also follows that

$$(u-5)^2 - (u-3)^2 - (u-1)^2 + (u+1)^2 - (u+3)^2 + (u+5)^2 + (u+7)^2 - (u+9)^2 = 0$$

for every $u \in \mathbb{Z}$; therefore, the number of representations of any integer n in the desired form is infinite.

□

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