MY SOLUTIONS TO PROBLEMS IN *MATHEMATICAL REFLECTIONS*

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O31 (Issue 6, 2006. Proposed by Jean-Charles Mathieux, Dakar University, Sénégal.) Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}^2.$$

Solution. Let $m \in \mathbb{N}$ with $n \geq m$. The desired identity is a special case of the more general result*

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{m} = \sum_{k=0}^{m} 2^k \binom{m}{k}\binom{n}{k}. \tag{1}$$

Indeed, letting $m = n$ in the identity in (1) we get

$$\begin{aligned}\sum_{k=0}^{n} 2^k \binom{n}{k}^2 &= \sum_{k=0}^{n} 2^k \binom{n}{k}\binom{n}{k} \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{n} \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{n+k-n} \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k},\end{aligned}$$

and we are done.

□

*For instance, see: H. S. Wilf. *generatingfunctionology*, 2nd edition, 1994, p. 127.

J39 (Issue 1, 2007. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Evaluate the product

$$(\sqrt{3}+\tan 1)(\sqrt{3}+\tan 2)\cdots(\sqrt{3}+\tan 29).$$

Solution. Let us set

$$\mathcal{A} := (\sqrt{3}+\tan 1)(\sqrt{3}+\tan 2)\cdots(\sqrt{3}+\tan 29). \tag{2}$$

Since $\sqrt{3}=\tan 60$, the equality in (2) can be rewritten as

$$\begin{aligned}\mathcal{A} &= (\sqrt{3}+\tan 1)(\sqrt{3}+\tan 2)\cdots(\sqrt{3}+\tan 29)\\ &= (\tan 60+\tan 1)(\tan 60+\tan 2)\cdots(\tan 60+\tan 29).\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{A} &= \left(\frac{\sin 60}{\cos 60}+\frac{\sin 1}{\cos 1}\right)\cdots\left(\frac{\sin 60}{\cos 60}+\frac{\sin 29}{\cos 29}\right)\\ &= \left(\frac{\sin 60\cos 1+\sin 1\cos 60}{\cos 60\cos 1}\right)\cdots\left(\frac{\sin 60\cos 29+\sin 29\cos 60}{\cos 60\cos 29}\right)\\ &= \left(\frac{\sin 61}{\cos 60\cos 1}\right)\cdots\left(\frac{\sin 89}{\cos 60\cos 29}\right).\end{aligned}$$

From the latter equality and the well known identity $\sin(90-x)=\cos x$, it follows that

$$\mathcal{A}=\left(\frac{1}{\cos 60}\right)^{29}=\left(\frac{1}{\frac{1}{2}}\right)^{29}=2^{29}.$$

□

U46 (Issue 2, 2007. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Let k be a positive integer and let

$$a_n = \left\lfloor (k+\sqrt{k^2+1})^n + \left(\frac{1}{2}\right)^n \right\rfloor, \quad n \geq 0.$$

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \frac{1}{8k^2}$.

Solution. Let us consider the sequence $\{b_m\}_{m\in\mathbb{Z}^+}$, defined for $m \geq 0$ as follows

$$\begin{cases} b_0 &= 2 \\ b_1 &= 2k \\ b_m &= 2k(b_{m-1}) + b_{m-2}. \end{cases}$$

Through generating functions we can establish that

$$b_n = (k+\sqrt{k^2+1})^n + (k-\sqrt{k^2+1})^n$$

for every $n \in \mathbb{Z}^+$. Now, since the inequalities

$$b_n \leq (k+\sqrt{k^2+1})^n + \left(\frac{1}{2}\right)^n < b_n + 1$$

hold for every $n \in \mathbb{Z}^+$, we conclude that

$$a_n = \left\lfloor (k+\sqrt{k^2+1})^n + \left(\frac{1}{2}\right)^n \right\rfloor = b_n,$$

for every nonnegative whole number n. Then,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} &= \sum_{n=1}^{\infty} \frac{1}{b_{n-1}b_{n+1}} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2k}\right)\left(\frac{1}{b_{n-1}b_n} - \frac{1}{b_nb_{n+1}}\right) \\ &= \frac{1}{2k}\sum_{n=1}^{\infty}\left(\frac{1}{b_{n-1}b_n} - \frac{1}{b_nb_{n+1}}\right). \end{aligned} \tag{3}$$

In order to reach the desired conclusion, we sum the telescoping series in the right-hand side of (3).

□

O47 (Issue 2, 2007. Proposed by Gabriel Alexander Reyes, San Salvador, El Salvador.) Consider the Fibonacci sequence, $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Prove that

$$\mathcal{S}_n := \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} F_k}{n+1-k} = \begin{cases} \frac{2F_{n+1}}{n+1} & \text{if } n \equiv 1 \pmod 2 \\ 0 & \text{if } n \equiv 0 \pmod 2). \end{cases}$$

Solution. Let $\alpha := \left(\frac{1+\sqrt{5}}{2}\right)$ and $\beta := \left(\frac{1-\sqrt{5}}{2}\right)$. Using these identifications we write Binet's formula as follows

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}.$$

The sum we wish to evaluate can now be rewritten as

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} F_k}{n+1-k} &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k}}{n+1-k} \left(\frac{\alpha^k - \beta^k}{\sqrt{5}} \right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} \alpha^k}{n+1-k} - \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} \beta^k}{n+1-k} \right). \qquad (4) \end{aligned}$$

In order to find a closed expression for $\mathcal{S}_n$, we first need to find a way to evaluate both sums in (4). It is toward this end that we introduce the auxiliary function

$$f(t) = (1-t)^n = \sum_{k=0}^{n} \binom{n}{k} (-t)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} t^{n-k}.$$

Integrating this function over the interval $[0, x]$ we get

$$\begin{aligned} \frac{1-(1-x)^{n+1}}{n+1} &= \frac{1}{n+1} - \frac{(1-x)^{n+1}}{n+1} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} x^{n-k+1}}{n+1-k} \\ &= x^{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} x^{-k}}{n+1-k}. \qquad (5) \end{aligned}$$

From (4) and (5), it follows that

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} F_k}{n+1-k} &= \frac{1}{\sqrt{5}} \left(\frac{\alpha^{n+1} - (\alpha-1)^{n+1}}{n+1} - \frac{\beta^{n+1} - (\beta-1)^{n+1}}{n+1} \right) \qquad (6) \\ &= \frac{\alpha^{n+1} - \beta^{n+1} + (\beta-1)^{n+1} - (\alpha-1)^{n+1}}{(n+1)\sqrt{5}}. \qquad (7) \end{aligned}$$

If n is even, we observe that

$$\begin{aligned} \alpha^{n+1} &= -(\beta-1)^{n+1} \\ (\alpha-1)^{n+1} &= -\beta^{n+1}. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} F_k}{n+1-k} &= \frac{\alpha^{n+1} - \beta^{n+1} + (\beta-1)^{n+1} - (\alpha-1)^{n+1}}{(n+1)\sqrt{5}} \\ &= 0, \end{aligned}$$

and the purported result holds in this case.

If n is odd,

$$\begin{aligned} \alpha^{n+1} &= (\beta - 1)^{n+1} \\ \beta^{n+1} &= (\alpha - 1)^{n+1}. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k} F_k}{n+1-k} &= \frac{\alpha^{n+1} - \beta^{n+1} + (\beta - 1)^{n+1} - (\alpha - 1)^{n+1}}{(n+1)\sqrt{5}} \\ &= \frac{2\alpha^{n+1} - 2\beta^{n+1}}{(n+1)\sqrt{5}} \\ &= \frac{2}{n+1} \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} \right) \\ &= \frac{2F_{n+1}}{n+1}, \end{aligned}$$

which implies that the assertion in question holds true in this case, too.

□

J55 (Issue 4, 2007. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Let $a_0 = 1$ and $a_{n+1} = a_0 \cdots a_n + 4$ for $n \geq 0$. Prove that $a_n - \sqrt{a_{n+1}} = 2$ for all $n \geq 1$.

Solution. We are going to proceed in a somewhat opposite direction. By means of mathematical induction we will show that the equality

$$a_{n+1} = (a_n - 2)^2 \tag{8}$$

holds for every $n \in \mathbb{N}$. This will in turn settle the original question.

The relation in (8) clearly holds if $n = 1$. Let us suppose that it also holds when n is an arbitrary natural number. Since

$$\begin{aligned} a_{n+2} &= a_0 \cdots a_n \cdot a_{n+1} + 4 \\ &= (a_0 \cdots a_n)(a_{n+1}) + 4 \\ &= (a_{n+1} - 4)(a_{n+1}) + 4 \\ &= a_{n+1}^2 - 4a_{n+1} + 4 \\ &= (a_{n+1} - 2)^2, \end{aligned}$$

it follows that (8) remains true for $n + 1$. Therefore, $a_{n+1} = (a_n - 2)^2$ for every $n \in \mathbb{N}$ and we are done.

□

O55 (Issue 4, 2007. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) For each positive integer k, let $f(k) = 4^k + 6^k + 9^k$. Prove that for all nonnegative integers m and n, $f(2^m)$ divides $f(2^n)$ whenever m is less than or equal to n.

Solution. We proceed by induction on n. If $n = 0$ or $n = 1$ the purported result clearly holds. Let us suppose that the claim remains true when n is a fixed yet arbitrary integer greater than 1. Since

$$\begin{aligned} f(2^{n+1}) &= 4^{2^{n+1}} + 6^{2^{n+1}} + 9^{2^{n+1}} \\ &= (4^{2^n} + 9^{2^n} + 6^{2^n})(4^{2^n} + 9^{2^n} - 6^{2^n}) \\ &= f(2^n)(4^{2^n} + 9^{2^n} - 6^{2^n}), \end{aligned}$$

it follows from the inductive hypothesis that $f(2^{n+1})$ is divided by $f(2^m)$ when $m = 0, 1, \ldots, n$. But trivially, it is also divided by $f(2^m)$ when $m = n + 1$. This culminates our demonstration.

□

U64 (Issue 5, 2007. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Let x be a real number. Define the sequence $\{x_n\}_{n=1}^{\infty}$ recursively by $x_1 = 1$, and $x_{n+1} = x^n + nx_n$ for $n \geq 1$. Prove that

$$\prod_{n=0}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right) = e^{-x}.$$

Solution. Our proof leans heavily on the following straightforward property of the sequence $\{x_n\}_{n=1}^{\infty}$: for all $n \in \mathbb{N}$

$$x_{n+1} = x^n + nx^{n-1} + n(n-1)x^{n-2} + \cdots + n!x + n! \tag{9}$$

Let us denote with $\mathcal{P}_k$ the k-th partial product of $\prod_{n=0}^{\infty}\left(1 - \frac{x^n}{x_{n+1}}\right)$. By means of of mathematical iduction we proceed to show that the equality

$$\begin{aligned} \mathcal{P}_k &= \left(1 - \frac{x}{x_2}\right)\left(1 - \frac{x^2}{x_3}\right)\cdots\left(1 - \frac{x^k}{x_{k+1}}\right) \\ &= \frac{1}{\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \cdots + \frac{x}{1!} + 1} \end{aligned} \tag{10}$$

holds true for every $k \in \mathbb{N}$. This is clearly the case for $k = 1$. Let us suppose the purported identity remains valid for k. Since,

$$\begin{aligned} \mathcal{P}_{k+1} &= \mathcal{P}_k\left(1 - \frac{x^{k+1}}{x_{k+2}}\right) \\ &= \left(\frac{1}{\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \cdots + \frac{x}{1!} + 1}\right)\left(1 - \frac{x^{k+1}}{x_{k+2}}\right) \\ &= \frac{x_{k+2} - x^{k+1}}{\left(\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \cdots + \frac{x}{1!} + 1\right)x_{k+2}}, \end{aligned}$$

the identity in (9) allows us to conclude that,

$$\begin{aligned} \mathcal{P}_{k+1} &= \frac{(k+1)x^k + (k+1)kx^{k-1} + \cdots + (k+1)!x + (k+1)!}{\left(\frac{x^k}{k!} + \frac{x^{k-1}}{(k-1)!} + \cdots + \frac{x}{1!} + 1\right)x_{k+2}} \\ &= \frac{(k+1)!}{x_{k+2}} \\ &= \frac{(k+1)!}{x^{k+1} + (k+1)x^k + \cdots + (k+1)!x + (k+1)!} \\ &= \frac{1}{\frac{x^{k+1}}{(k+1)!} + \frac{x^k}{k!} + \cdots + \frac{x}{1!} + 1} \end{aligned}$$

as desired. It follows from (9) and (10) that,

$$\begin{aligned}
\prod_{n=0}^{\infty}\left(1-\frac{x^n}{x_{n+1}}\right) &= \prod_{n=1}^{\infty}\left(1-\frac{x^n}{x_{n+1}}\right) \\
&= \lim_{k\to\infty}\mathcal{P}_k \\
&= \lim_{k\to\infty}\frac{1}{\frac{x^k}{k!}+\frac{x^{k-1}}{(k-1)!}+\cdots+\frac{x}{1!}+1} \\
&= \frac{1}{\lim_{k\to\infty}\left(\frac{x^k}{k!}+\frac{x^{k-1}}{(k-1)!}+\cdots+\frac{x}{1!}+1\right)} \\
&= \frac{1}{\sum_{k=0}^{\infty}\frac{x^k}{k!}} \\
&= \frac{1}{e^x} \\
&= e^{-x}
\end{aligned}$$

and we are done.

□

J66 (Issue 5, 2007. Proposed by Ivan Borsenco, UT at Dallas, USA.) Let $a_0 = a_1 = 1$ and $a_{n+1} = 2a_n - a_{n-1} + 2$ for $n \geq 1$. Prove that $a_{n^2+1} = a_{n+1}a_n$ for all $n \geq 0$.

Solution. The generatingfunctionological approach allows us to establish the purported result in a single swoop: the sequence $\{a_n\}_{n=0}^{\infty}$ is such that

$$a_n = n(n-1) + 1$$

for all $n \in \mathbb{Z}^+$. Therefore,

$$\begin{aligned} a_{n+1}a_n &= [(n+1)n+1][n(n-1)+1] \\ &= n^2(n+1)(n-1) + (n+1)n + (n-1)n + 1 \\ &= n^2(n^2-1) + 2n^2 + 1 \\ &= n^2(n^2+1) + 1 \\ &= a_{n^2+1}, \end{aligned}$$

and we are done.

□

J67 (Issue 6, 2007. Proposed by Ivan Borsenco, UT at Dallas, USA.) Prove that among seven arbitrary perfect squares there are two whose difference is divisible by 20.

Solution. There are only 6 possibilities for the residue of a perfect square upon division by 20. Therefore, in any set of seven arbitrary perfect squares, we can always find two elements a and b that *live* in the same residue class modulo 20. This fact implies that $a \equiv b \pmod{20}$ and we are done.

□

J71 (Issue 6, 2007. Proposed by Samin Riasat, Notre Dame College, Dakha, Bangladesh.) In the Cartesian plane call a line *good* if it contains infinitely many lattice points. Two lines intersect at a lattice point at an angle of 45°. Prove that if one of the lines is *good*, then so is the other.

Solution. Let us suppose that l_1 and l_2 are two lines that satisfy the conditions stated in the hypothesis. Without loss of generality we may assume that l_1 is a *good* line and that the coordinates of the lattice point at which those lines meet are (m, n).

The purported result holds trivially in any one of the following cases:

(1) The slope of line l_1 is 1.
(2) The slope of line is l_1 is -1.
(3) Line l_1 is vertical.
(4) Line l_1 is horizontal.

If line l_1 falls into neither of the categories above, we infer that its slope is a rational number $\frac{a}{b}$, where $a, b \in \mathbb{Z}\backslash\{0\}$, $a + b \neq 0$, and $a - b \neq 0$. Furthermore, the hypothesis that lines l_1 and l_2 intersect at an angle of 45° imply that one and only one of the relations below is satisfied

$$\alpha_2 = \alpha_1 + 45 \tag{11}$$

$$\alpha_2 = \alpha_1 - 45. \tag{12}$$

(Here we have denoted with α_1 and α_2 the elevation angles of lines l_1 and l_2 respectively.)

Now, assuming the validity of (11), the well-known identity

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

allows us to conclude that

$$\begin{aligned} \tan \alpha_2 &= \tan(\alpha_1 + 45) \\ &= \frac{\tan \alpha_1 + \tan 45}{1 - \tan \alpha_1 \tan 45} \\ &= \frac{\frac{a}{b} + 1}{1 - \frac{a}{b}} \\ &= \frac{a + b}{b - a}. \end{aligned}$$

Hence, line l_2 is represented by equation

$$\begin{aligned} y - n &= \tan \alpha_2 (x - m) \\ &= \left(\frac{a + b}{b - a}\right)(x - m), \end{aligned}$$

or equivalently,

$$(b - a)y - (a + b)x = (b - a)n - (a + b)m. \tag{13}$$

Since $\gcd(b - a, -(a + b)) \mid (b - a)n - (a + b)m$, the (underlying) diophantine equation in (13) possesses an infinite number of solutions in integers. Each one of these solutions corresponds with a lattice point in l_2 and we are done. The result is established analogously if we assume the validity of the relation in (12) instead.

□

J73 (Issue 1, 2008. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Let

$$a_n = \begin{cases} n^2 - n & \text{if 4 divides } n^2 - n \\ n - n^2 & \text{otherwise.} \end{cases}$$

Evaluate $a_1 + a_2 + \cdots + a_{2008}$.

Solution. $4 \mid n^2 - n = n(n-1) \Leftrightarrow n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. Therefore,

$$\begin{aligned}
a_1 + a_2 + \cdots + a_{2008} &= \sum_{k=1}^{2008} a_k \\
&= \sum_{k=1}^{502} a_{4k} + \sum_{k=0}^{501} a_{4k+1} + \sum_{k=0}^{501} a_{4k+2} + \sum_{k=0}^{501} a_{4k+3} \\
&= \sum_{k=0}^{501} a_{4(k+1)} + \left(\sum_{k=0}^{501} a_{4k+1} + \sum_{k=0}^{501} a_{4k+2} \right) + \sum_{k=0}^{501} a_{4k+3} \\
&= \sum_{k=0}^{501} 4(k+1)(4k+3) + \sum_{k=0}^{501} (4k+1)(-2) - \sum_{k=0}^{501} (4k+3)(4k+2) \\
&= \sum_{k=0}^{501} 4(k+1)(4k+3) - \sum_{k=0}^{501} (4k+3)(4k+2) + \sum_{k=0}^{501} (4k+1)(-2) \\
&= \sum_{k=0}^{501} (4k+3)(2) + \sum_{k=0}^{501} (4k+1)(-2) \\
&= \sum_{k=0}^{501} (2)(4k+3-4k-1) \\
&= \sum_{k=0}^{501} 4 \\
&= 2008,
\end{aligned}$$

and we are done.

□

U85 (Issue 3, 2008. Proposed by Brian Bradie, Christopher Newport University, USA.) Evaluate

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1^3+\cdots+k^3}$

b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{1^3+\cdots+k^3}$

Solution.

a) From

$$\frac{1}{k^2(k+1)^2} = \left(\frac{2}{k+1}-\frac{2}{k}\right)+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

and the fact that $\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k}\right) = -1$ and $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$, we conclude that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{1^3+2^3+\cdots+k^3} &= 2^2\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)^2} \\
&= 2^2\left\{2\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k}\right)+\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\right\} \\
&= 2^2\left(2(-1)+\frac{\pi^2}{6}+\left(\frac{\pi^2}{6}-1\right)\right) \\
&= 2^2\left(\frac{\pi^2}{3}-3\right) \\
&= \frac{4(\pi^2-9)}{3}.
\end{aligned}$$

b) The series development for $\ln 2$ is crucial here. Indeed,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{1^3+2^3+\cdots+k^3} &= 2^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2(k+1)^2} \\
&= 2^2\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{2}{k+1}-\frac{2}{k}+\frac{1}{k^2}+\frac{1}{(k+1)^2}\right) \\
&= 2^2\cdot 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+1}-2^2\cdot 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} \\
&+ 2^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}+2^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+1)^2} \\
&= 2^2\cdot 2(1-\ln 2)-2^2\cdot 2(\ln 2) \\
&+ 2^2\left(\frac{\pi^2}{12}\right)+2^2\left(1-\frac{\pi^2}{12}\right) \\
&= 4(3-4\ln 2).
\end{aligned}$$

□

J94 (Issue 4, 2008. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Prove that the equation $x^3 + y^3 + z^3 + w^3 = 2008$ has infinitely many solutions in integers.

Solution. For every $n \in \mathbb{Z}$, the 4-tuple

$$(x = 10 + 60n^3, \quad y = 10 - 60n^3, \quad z = 2, \quad w = -60n^2)$$

provides us with a solution to the given equation. Since $n^3 = m^3$ implies $n = m$, we have that no two of these solutions are identical: this terminates our proof.

□

S115 (Issue 2, 2009. Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania.) Prove that for each positive integer n, 2009^n can be written as a sum of six nonzero perfect squares.

Solution. Since

$$2009 = 7^2 \cdot 41 = 7^2(6^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2), \tag{14}$$

the claim holds true when $n = 1$. This implies in turn the veracity of the claim whenever n is odd. Indeed, if $n = 2k + 1$ then

$$2009^n = 2009^{2k+1} = 2009^{2k} \cdot 2009 = (2009^k)^2 \cdot 2009.$$

It remains to establish the result for even n. We proceed by induction now. In the light of (14), we have that

$$\begin{aligned} 2009^2 &= [(42^2 + 7^2 + 7^2 + 7^2) + (7^2 + 7^2)]^2 \\ &= (42^2 + 7^2 + 7^2 + 7^2)^2 + 2(42^2 + 7^2 + 7^2 + 7^2)(7^2 + 7^2) + (7^2 + 7^2)^2. \end{aligned}$$

Since

$$\begin{aligned} 2(42^2 + 7^2 + 7^2 + 7^2)(7^2 + 7^2) &= 2(42^2 + 7^2)(7^2 + 7^2) + 2(7^2 + 7^2)(7^2 + 7^2) \\ &= 4 \cdot 7^2(42^2 + 7^2) + 4 \cdot 7^2(7^2 + 7^2) \end{aligned}$$

we conclude that $2(42^2 + 7^2 + 7^2 + 7^2)(7^2 + 7^2)$ can be written as a sum of four nonzero perfect squares. Thus, 2009^2 is a sum of six nonzero perfect squares and this establishes the validity of the basis for the induction.

Let us suppose now that 2009^{2k} can be written as a sum of six nonzero perfect squares. This supposition implies at once that

$$2009^{2(k+1)} = 2009^{2k+2} = 2009^{2k} \cdot 2009^2$$

is a sum of six nonzero perfect squares and the proof terminates.

□

U157 (Issue 3, 2010. Proposed by Mihai Piticari, Dragos Voda National College, Campulung Moldovenesc, Romania.) Let $(A, +, \cdot)$ be a finite ring such that $1 + 1 = 0$. Prove that the number of solutions to the equation $x^2 = 0$ is equal to the number of solutions to the equation $x^2 = 1$.

Solution. The problem is equivalent to finding a bijection between the sets

$$X = \{x \in A : x^2 = 0\} \quad \text{and} \quad Y = \{x \in A : x^2 = 1\}.$$

We claim that the function $f : X \to Y$ given by $f(x) = 1 - x$ does the job. The injectivity of f is clear. In order to prove its surjectivity, it suffices to establish that if $x \in Y$ then $1 - x \in X$. This can in fact be guaranteed by the following equalities

$$(1 - x)^2 = 1 - x - x + x^2 = 1 - (1 + 1)x + 1 = (1 + 1)(1 - x) = 0$$

and our proof terminates.

□

Scholium. Our proof shows that the finiteness of A is a hypothesis with which we can dispense.

J159 (Issue 3, 2010. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Find all integers n for which $9n + 16$ and $16n + 9$ are both perfect squares.

Solution. If $9n + 16$ and $16n + 9$ are both perfect squares then $n \geq 0$ and the number $p_n = (9n + 16)(16n + 9) = (12n)^2 + (9^2 + 16^2)n + 12^2$ is also a perfect square. Since

$$(12n + 12)^2 \leq (12n)^2 + (9^2 + 16^2)n + 12^2 < (12n + 15)^2$$

it follows that if $n > 0$ then we must have $p_n = (12n + 13)^2$ or $p_n = (12n + 14)^2$. The former condition gives $n = 1$ and the latter, $n = 52$. Therefore, $n = 0$, $n = 1$, and $n = 52$ are the only integers n for which the expressions $9n + 16$ and $16n + 9$ simultaneously return perfect squares.

□

U188 (Issue 2, 2011. Proposed by Roberto Bosch Cabrera, Florida, USA.) Let G be a finite group in which for every positive integer m the number of solutions in G of the equation $x^m = e$ is at most m. Prove that G is cyclic.

Solution. Let n be the order of G. For every $d \in \mathbb{N}$, let us denote with A_d the subset of G that consists of all elements that have order d. Now, if $f(d) := |A_d|$, we affirm that $f(d) \leq \phi(d)$ for every $d \in \mathbb{N}$, where ϕ is the totient function of Euler. Indeed, if $f(d) = 0$, the purported inequality follows from the possitivity of the totient function. Otherwise, fix a $g \in G$ with order d. It follows that all elements of G of order d belong to the set $\{1, g, g^2, \ldots, g^{d-1}\}$. Since g^i has order d iff $(d, i) = 1$, it is straightforward to conclude that $f(d) = \phi(d)$ in this case ($f(d) \neq 0$, that is).

Further, the identities $n = |G| = |\bigsqcup_{d|n} A_d| = \sum_{d|n} |A_d| = \sum_{d|n} f(d)$ and $n = \sum_{d|n} \phi(d)$ imply at once that none of the inequalities $f(d) \leq \phi(d)$ can be strict when d is a divisor of n. In particular, this implies that

$$|A_n| = f(n) = \phi(n) > 0$$

and we are done.

□

J191 (Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Find all positive integers $n \geq 2$ for which

$$(n-2)! + (n+2)!$$

is a perfect square.

Solution. Let $a_n := (n-2)! + (n+2)!$. Since

$$\begin{aligned} a_n &= (n-2)![1 + (n-1)n(n+1)(n+2)] \\ &= (n-2)![1 + (n^2+n-2)(n^2+n)] \\ &= (n-2)![1 + (n^2+n-1-1)(n^2+n-1+1)] \\ &= (n-2)!(n^2+n-1)^2, \end{aligned}$$

it follows that a_n is a perfect square iff $(n-2)!$ is a perfect square. Then, the problem reduces to determining all natural numbers N such that $N!$ is a perfect square.

Let N be a natural number greater than 1. If

$$p = \max\{p \in [1, N] : p \text{ is a prime number}\},$$

we ascertain that $N < 2p$. Indeed, if this were not the case, we would have $p < 2p \leq N$. Bertrand's Postulate would yield in that scenario a prime q such that $p < q \leq 2p \leq N$ (impossible!). Therefore, $N < 2p$ and the exponent to which p appears in the canonical factorization of N is 1. This implies in turn that $N!$ can't be a perfect square whenever that $N > 1$.

Hence, a_n is a perfect square iff $(n-2)!$ is a perfect square iff $n \in \{2, 3\}$. □

U195 (Issue 3, 2011. Proposed by Roberto Bosch Cabrera, Florida, USA.) Given a positive integer n, let $f(n)$ be the square of the number of its digits. For example, $f(2) = 1$ and $f(123) = 9$. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)}$$

converges.

Solution. Let k be a fixed but arbitrary natural number. We know that, if m_k is the number of digits in the decimal expansion of k, then $10^{m_k-1} \leq k < 10^{m_k}$. It follows that

$$\begin{aligned}
\sum_{n=1}^{k} \frac{1}{nf(n)} &= \sum_{n=1}^{10-1} \frac{1}{nf(n)} + \sum_{n=10}^{10^2-1} \frac{1}{nf(n)} + \cdots + \sum_{n=10^{m_k-1}}^{k} \frac{1}{nf(n)} \\
&= \sum_{n=1}^{10-1} \frac{1}{n \cdot 1^2} + \sum_{n=10}^{10^2-1} \frac{1}{n \cdot 2^2} + \cdots + \sum_{n=10^{m_k-1}}^{k} \frac{1}{n \cdot m_k^2} \\
&\leq \sum_{n=1}^{10-1} \frac{1}{1 \cdot 1^2} + \sum_{n=10}^{10^2-1} \frac{1}{10 \cdot 2^2} + \cdots + \sum_{n=10^{m_k-1}}^{k} \frac{1}{10^{m_k-1} \cdot m_k^2} \\
&< \frac{10}{1 \cdot 1^2} + \frac{10^2}{10 \cdot 2^2} + \cdots + \frac{10^{m_k}}{10^{m_k-1} \cdot m_k^2} \\
&= 10 \times \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m_k^2} \right) \\
&\leq 10 \cdot \zeta(2)
\end{aligned}$$

and we are done.

□

U206 (Issue 5, 2011. Proposed by Gabriel Dospinescu, École Polytechnique, France.) Prove that there is precisely one group with 30 elements and 8 automorphisms.

Solution. We first show that no non-abelian group of order 30 has automorphism group of order 8. The main ingredients of this part of our solution are two well-known lemmata from Group Theory, namely:

a) If $\mathbf{Z}(G)$ is the center of G then, $G/\mathbf{Z}(G)$ cylic implies G abelian, and
b) $G/\mathbf{Z}(G) \cong \mathrm{Inn}(G)$, the group of inner automorphisms of G.

From a) we derive that $|\mathbf{Z}(G)| \in \{1, 3, 5\}$. If $|\mathbf{Z}(G)| = 1$ or $|\mathbf{Z}(G)| = 3$, the claim in b) allows us to acertain that the automorphism group of G has order 10 (at the very least). If $|\mathbf{Z}(G)| = 5$, the lemma in b) indicates that the automorphism group of G has for order a multiple of 6. In any case, we do have that the automorphism group of G does not have order 8.

On the other hand, we ascertain that every abelian group G of order 30 is cyclic: by Cauchy's theorem, we always have elements $a, b, c \in G$ of orders 2, 3, and 5, respectively. According to standard group-theoretical lore, abc has order 30 in that case and whence $G = \langle abc \rangle$. Since any two cyclic groups of the same order are isomorphic and $|\mathrm{Aut}(\mathbb{Z}/30\mathbb{Z})| = \phi(30) = 8$, it follows that, up to isomorphism, $\mathbb{Z}/30\mathbb{Z}$ is the only group of order 30 with 8 automorphisms.

□

S247 (Issue 6, 2012. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Prove that for any positive integers m and n, the number $8m^6 + 27m^3n^3 + 27n^6$ is composite.

Solution. Since

$$(2m^2 + 3n^2)^3 = 8m^6 + 36m^4n^2 + 54m^2n^4 + 27n^6,$$

it follows that

$$\begin{aligned} 8m^6 + 27m^3n^3 + 27n^6 &= (2m^2 + 3n^2)^3 - (36m^4n^2 + 54m^2n^4) + 27m^3n^3 \\ &= (2m^2 + 3n^2)^3 - 18m^2n^2(2m^2 + 3n^2) + 27m^3n^3. \end{aligned}$$

Then, letting $A := 2m^2 + 3n^2$ and $B := 3mn$, the expression in the previous line becomes

$$\begin{aligned} A^3 - 2AB^2 + B^3 &= A^3 - AB^2 - AB^2 + B^3 \\ &= A(A - B)(A + B) - B^2(A - B) \end{aligned}$$

Therefore, $2m^2+3n^2-3mn$ is always a divisor of $8m^6+27m^3n^3+27n^6$. In addition, the assumption that both m and n are positive integers allows us to ascertain at once that $2m^2 + 3n^2 - 3mn$ is a proper divisor of $8m^6 + 27m^3n^3 + 27n^6$. Indeed, $3mn > 0$ and whence

$$2m^2 + 3n^2 - 3mn < 2m^2 + 3n^2 < 8m^6 + 27m^3n^3 + 27n^6.$$

On the other hand,

$$2m^2 + 3n^2 - 3mn = (m - n)^2 + (m - n)^2 + n^2 + mn > 1.$$

Done.

□

S265 (Issue 3, 2013. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Find all pairs (m, n) of positive integers such that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares.

Solution. If $m < n$ then

$$n^2 < n^2 + 5m < n^2 + 5n < (n + 3)^2.$$

Therefore, either $n^2 + 5m = (n + 1)^2$ or $n^2 + 5m = (n + 2)^2$. In the first case, we infer that $n = 5k + 2$ for some $k \in \mathbb{N}$. Then, $m^2 + 5n = (2k + 1)^2 + 5(5k + 2)$. The hypothesis that this number is a perfect square and the inequalities

$$(2k + 4)^2 < (2k + 1)^2 + 5(5k + 2) = 4k^2 + 29k + 11 < (2k + 8)^2$$

imply in turn that $k \in \{5, 38\}$. If $k = 5$, then $m = 11$ and $n = 27$; if $k = 38$, then $m = 77$ and $n = 192$. Both of the associated pairs $(m = 11, n = 27)$ and $(m = 77, n = 192)$ satisfy that $m^2 + 5n$ and $n^2 + 5m$ are perfect squares. In the second case, we infer that $n = 5k - 1$ for some $k \in \mathbb{N} \setminus \{1\}$. Then, $m^2 + 5n = (4k)^2 + 5(5k - 1)$. The hypothesis that this number is a perfect square and the inequalities

$$16k^2 < (4k)^2 + 5(5k - 1) = 16k^2 + 25k - 5 < (4k + 4)^2$$

imply in turn that $k = 14$, $m = 56$, and $n = 69$. The pair $(m = 56, n = 69)$ does satisfy that $m^2 + 5n$ and $n^2 + 5m$ are perfect squares.

If $m = n$, then $m^2 < m^2+5n = m^2+5m < (m+3)^2$. Hence, $m^2+5m = (m+1)^2$ or $m^2+5m = (m+2)^2$. The former equation does not have solutions in $\mathbb{N}$ and the latter implies that $m = 4 = n$. The associated pair (m, n) is $(4, 4)$, which clearly satisfies that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares.

Finally, since the existence of a pair (m, n) with $m > n$ such that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares implies the existence of a pair (m', n'), with $m' < n'$, which also satisfies the two constraints under consideration, we conclude that there exactly seven pairs $(m, n) \in \mathbb{N}$ such that $m^2 + 5n$ and $n^2 + 5m$ are both perfect squares:

$$(11, 27), (56, 69), (77, 192), (4, 4), (27, 11), (69, 56), (192, 77).$$

□

J299 (Issue 2, 2014. Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA.) Prove that no matter how we choose n numbers from the set $\{1, 2, \dots, 2n\}$, one of them will be a square-free integer.

Solution. It suffices to show that, for every $n \in \mathbb{N}$, the number of natural numbers in the interval $[1, 2n]$ which are not square-free is less than n. Since the thesis of the problem is trivially true for $n = 1$, in what follows we suppose that $n > 1$.

Let us denote by $\overline{Q}_{2n}$ the set of natural numbers in the interval $[1, 2n]$ which are not square-free. Besides, if $p \in [1, 2n]$ is a prime a number, let us denote by $\mathcal{M}(p^2)$ the set of multiples of p^2 which belong to the interval $[1, 2n]$. So, if q is the greatest prime number in $[1, \sqrt{2n}]$ then

$$\overline{Q}_{2n} = \mathcal{M}(2^2) \cup \mathcal{M}(3^2) \cup \mathcal{M}(5^2) \cup \ldots \cup \mathcal{M}(q^2). \tag{15}$$

Since any natural number N has $\left\lfloor \frac{2n}{N} \right\rfloor$ multiples in the interval $[1, 2n]$, we obtain from (15) that

$$\begin{aligned} |\overline{Q}_{2n}| &\leq \left\lfloor \frac{2n}{2^2} \right\rfloor + \left\lfloor \frac{2n}{3^2} \right\rfloor + \left\lfloor \frac{2n}{5^2} \right\rfloor + \cdots + \left\lfloor \frac{2n}{q^2} \right\rfloor \\ &\leq \frac{2n}{2^2} + \frac{2n}{3^2} + \frac{2n}{5^2} + \cdots + \frac{2n}{q^2} \\ &= 2n \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{q^2} \right) \\ &< 2n \sum_p \frac{1}{p^2} \qquad (16) \end{aligned}$$

where $\sum_p \frac{1}{p^2}$ is the series of the squared reciprocals of the prime numbers. This series is convergent and the well-known fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ implies that

$$\begin{aligned} \frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &> 1 + \sum_p \frac{1}{p^2} + \sum_{n=2}^{\infty} \frac{1}{(2n)^2} \\ &= 1 + \sum_p \frac{1}{p^2} + \frac{1}{4} \left(\frac{\pi^2}{6} - 1 \right) \\ &= \frac{3}{4} + \frac{\pi^2}{24} + \sum_p \frac{1}{p^2}. \qquad (17) \end{aligned}$$

From (16) and (17) we conclude that

$$|\overline{Q}_{2n}| < 2n \left(\frac{\pi^2}{8} - \frac{3}{4} \right) = n \left(\frac{\pi^2 - 6}{4} \right) < n$$

which was what we desired to establish.

□

J331 (Issue 2, 2015. Proposed by Alessandro Ventullo, Milan, Italy.) Determine all positive integers n such that

$$n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n!}\right)$$

is divisible by n.

Solution. Our solution relies on the criterion for primality given by Wilson's Theorem and its converse: *if $m > 1$ is a natural number, then a necessary and sufficient condition that m should be prime is that* $(m-1)!+1 \equiv 0 \pmod{m}$. Since this result is well-known and its proof can be found in many an introductory text on Number Theory (see, for instance: G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers.* Sixth Ed., Oxford University Press, 2008, pp. 85–86.), we do not prove it here and proceed instead to settle the problem under consideration.

Firstly, let us suppose that $n > 4$. Since

$$\begin{aligned} n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n!}\right) &= n!+n(n-1)\cdots 3\cdot 1+\cdots+n(n-2)! \\ &+ (n-1)!+1, \end{aligned}$$

it follows that

$$n \mid n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n!}\right)$$

if and only if

$$(n-1)!+1 \equiv 0 \pmod{n}. \tag{18}$$

From the aforementioned criterion and (18) we obtain that, in the range $n > 4$, the condition

$$n \mid n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n!}\right)$$

holds if and only if n is a prime number. This fact and the following calculations

$$\begin{aligned} 1!\left(1+\frac{1}{1!}\right) &= 1\cdot 2 \\ 2!\left(1+\frac{1}{2}+\frac{1}{2!}\right) &= 2\cdot 2 \\ 3!\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3!}\right) &= 3\cdot 4 \\ 4!\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{4!}\right) &= 51 \end{aligned}$$

allow us to conclude a positive integer n satisfies the condition

$$n \mid n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n!}\right)$$

if and only if $n = 1$ or n is a prime number.

□

J337 (Issue 3, 2015. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Prove that for each integer $n \geq 0$, $16^n + 8^n + 4^{n+1} + 2^{n+1} + 4$ has two factors greater than 4^n.

Solution. If we let $x = 2^n$, then the expression $16^n + 8^n + 4^{n+1} + 2^{n+1} + 4$ becomes

$$\begin{aligned} x^4 + x^3 + 4x^2 + 2x + 4 &= (x^4 + 4x^2 + 4) + (x^3 + 2x) \\ &= (x^2 + 2)^2 + x(x^2 + 2) \\ &= (x^2 + 2)(x^2 + x + 2). \end{aligned}$$

It follows that, for every $n \in \mathbb{Z}^+$,

$$16^n + 8^n + 4^{n+1} + 2^{n+1} + 4 = (4^n + 2)(4^n + 2^n + 2).$$

Since the numbers $4^n + 2$ and $4^n + 2^n + 2$ are both clearly greater than 4^n, our proof terminates.

□

J339 (Issue 3, 2015. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Solve in positive integers the equation

$$\frac{x-1}{y+1}+\frac{y-1}{z+1}+\frac{z-1}{x+1}=1. \tag{19}$$

Solution. Let us suppose that x, y, and z are positive integers which satisfy the equation under consideration and that $z \geq x, y$. It follows that

$$\frac{x-1}{z+1} \leq \frac{x-1}{y+1}, \quad \frac{y-1}{z+1} \leq \frac{y-1}{z+1}, \quad \frac{z-1}{z+1} \leq \frac{z-1}{x+1},$$

and whence

$$\frac{(x-1)+(y-1)+(z-1)}{z+1} \leq 1.$$

The latter inequality implies at once that $x+y \leq 4$. Hence, we have to ponder the following six possibilities:

A. $x = 1$ and $y = 3$. In this case equation (19) becomes $\frac{2}{z+1}+\frac{z-1}{2}=1$, which can be rewritten as $(z-1)^2 = 0$ and wherefrom we get that $z = 1$.

B. $x = 1$ and $y = 2$. In this case equation (19) becomes $\frac{1}{z+1}+\frac{z-1}{2}=1$, which can be rewritten as $z^2-2z-1 = 0$ and wherefrom we can see that $z \notin \mathbb{N}$.

C. $x = 1$ and $y = 1$. In this case equation (19) becomes $\frac{z-1}{2}=1$, whose solution is $z = 3$.

D. $x = 2$ and $y = 2$. In this case equation (19) becomes $\frac{1}{3}+\frac{1}{z+1}+\frac{z-1}{3}=1$, which can be rewritten as $z(z-2) = 0$ and wherefrom we get that $z = 0$ and $z = 2$.

E. $x = 2$ and $y = 1$. In this case equation (19) becomes $\frac{1}{2}+\frac{z-1}{3}=1$, whose solution is $z = \frac{5}{2}$.

F. $x = 3$ and $y = 1$. In this case equation (19) becomes $1+\frac{z-1}{4}=1$, whose solution is $z = 1$.

The values obtained for z in A, B, E, and F are inadmissible (either we don't get a positive integral value for z in those cases or the constraint $z \geq x, y$ is not satisfied); therefore, the solutions to the original equation (subject to the additional constraint $z \geq x, y$) are: $x = 1, y = 1, z = 3$ and $x = y = z = 2$. If we drop the constraint $z \geq x, y$ we obtain the rest of solutions to equation (19), namely: $x = 3, y = 1, z = 1$ and $x = 1, y = 3, z = 1$.

□

J343 (Issue 4, 2015. Proposed by Titu Andreescu, University of Texas at Dallas, USA.) Prove that the number $10240\ldots002401$, having a total of 2014 zeros, is composite.

Solution. Firstly, we note that

$$\begin{aligned} 102400\ldots002401 &= 1024 \cdot 10^{2016} + 2401 \\ &= 2^{10} \cdot 10^{2016} + 7^4 \\ &= 7^4 + 4(2^2 \cdot 10^{504})^4. \end{aligned} \tag{20}$$

Next, as a consequence of the Sophie Germain Identity

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab),$$

we get that

$$d = 7^2 + 2(2^2 \cdot 10^{504})^2 + 2(7 \cdot 2^2 \cdot 10^{504})$$

is a divisor of $102400\ldots002401$. Since $d \in (1, 102400\ldots002401)$, the assertion under consideration follows.

□

U348 (Issue 4, 2015. Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Spain.) Evaluate the linear integral

$$\oint_c \frac{(1+x^2-y^2)\,dx + 2xy\,dy}{(1+x^2-y^2)^2+4x^2y^2}$$

where c is the square [sic] with vertices $(2,0)$, $(2,2)$, $(-2,2)$, and $(-2,0)$ traversed counterclockwise.

Solution. The integral under consideration is equal to the real part of the following complex integral

$$\oint_c \frac{1}{z^2+1}\,dz. \tag{21}$$

Indeed, if we write z in the form $x+iy$, where $x,y\in\mathbb{R}$, then

$$\begin{aligned}
\operatorname{Re}\left(\oint_c \frac{1}{z^2+1}\,dz\right) &= \operatorname{Re}\left(\oint_c \frac{1}{(1+x^2-y^2)+2ixy}\,(dx+idy)\right)\\
&= \operatorname{Re}\left(\oint_c \frac{(1+x^2-y^2)-2ixy}{(1+x^2-y^2)^2+4x^2y^2}\,(dx+idy)\right)\\
&= \oint_c \frac{(1+x^2-y^2)\,dx+2xy\,dy}{(1+x^2-y^2)^2+4x^2y^2}.
\end{aligned}$$

Thus, in order to solve the problem it suffices to evaluate the integral in (21). This can be done by means of the Cauchy Integral Formula. Let U be any region of the complex plane containing c. If $-i\in\mathbb{C}\setminus U$, then the function $f\colon U\to\mathbb{C}$ determined by the assignation $z\to\frac{1}{z+i}$ is holomorphic on U; the Cauchy Integral Formula gives us in this case that

$$\oint_c \frac{1}{z^2+1}\,dz = \oint_c \frac{\frac{1}{z+i}}{z-i}\,dz = 2\pi i f(i) = \pi.$$

Hence,

$$\oint_c \frac{(1+x^2-y^2)\,dx+2xy\,dy}{(1+x^2-y^2)^2+4x^2y^2} = \operatorname{Re}\left(\oint_c \frac{1}{z^2+1}\,dz\right) = \pi$$

and we are done.

□

O361 (Issue 1, 2016. Proposed by Alessandro Ventullo, Milan, Italy.) Determine the least integer $n > 2$ such that there are n consecutive integers whose sum of squares is a perfect square.

Solution. We claim that the first such n is 11. In order to prove this we first show that $n \notin \{3, 4, 5, \ldots, 10\}$:

- Since a perfect square is congruent to 0 or 1 modulo 3, it follows that $\sum_{j=1}^{3}(k+j)^2 \equiv 2 \pmod 3$ for every $k \in \mathbb{Z}$; hence, n cannot be equal to 3.
- Since a perfect square is congruent to 0 or 1 modulo 4, it follows that $\sum_{j=1}^{4}(k+j)^2 \equiv 2 \pmod 4$ for every $k \in \mathbb{Z}$; therefore, n cannot be equal to 4. Furthermore, the congruence $\sum_{j=1}^{4}(k+j)^2 \equiv 2 \pmod 4$ implies that
$$\sum_{j=1}^{5}(k+j)^2 \equiv 2 \pmod 4 \quad \text{or} \quad \sum_{j=1}^{5}(k+j)^2 \equiv 3 \pmod 4,$$
which indicates that n cannot be equal to 5.
- Since a perfect square is congruent to 0, 1 or 4 modulo 8, it follows that either $\sum_{j=1}^{6}(k+j)^2 \equiv 3 \pmod 8$ or $\sum_{j=1}^{6}(k+j)^2 \equiv 7 \pmod 8$ for every $k \in \mathbb{Z}$; hence, n cannot be equal to 6.
- The sum of the squares of 7 consecutive integers is equal to $\sum_{j=1}^{7}(k+j)^2 = 7(k^2+8k+20)$ for some $k \in \mathbb{Z}$. On the one hand, it is easy to see that, in order for $7(k^2+8k+20)$ to be a perfect square, 7 has to be a divisor of $k^2+8k+20$; on the other hand, we claim that it cannot be the case that $7 \mid k^2+8k+20$ for some $k \in \mathbb{Z}$: indeed, if 7 divided $k^2+8k+20$ for some $k \in \mathbb{Z}$, it would follow that $(k+4)^2 \equiv -4 \pmod 7$ which would imply in turn that $1 = (\frac{-4}{7}) = (\frac{-1}{7}) = (-1)^{\frac{7-1}{2}} = -1$, an absurdity! It follows from all this that n cannot be equal to 7.
- The sum of the squares of 8 consecutive integers is equal to $\sum_{j=1}^{8}(k+j)^2 = 4(2k^2+18k+51)$ for some $k \in \mathbb{Z}$. Since a perfect square is congruent to 0 or 1 modulo 4, but
$$2k^2+18k+51 \equiv 3 \pmod 4$$
for every $k \in \mathbb{Z}$, it follows that n cannot be equal to 8.
- The sum of the squares of 9 consecutive integers is equal to $\sum_{j=1}^{9}(k+j)^2 = 3(3k^2+30k+95)$ for some $k \in \mathbb{Z}$. Since
$$3k^2+30k+95 \equiv 2 \pmod 3$$
for every $k \in \mathbb{Z}$, it follows that there doesn't exist $k \in \mathbb{Z}$ such that $3 \mid 3k^2+30k+95$; therefore, $3(3k^2+30k+95)$ is never a perfect square. Thus, n cannot be equal to 9.
- The sum of the squares of 10 consecutive integers is equal to $\sum_{j=1}^{10}(k+j)^2 = 5(2k^2+22k+77)$ for some $k \in \mathbb{Z}$. If $5(2k^2+22k+77)$ were a perfect square for some $k \in \mathbb{Z}$, then
$$2k^2+22k+77 \equiv 0 \pmod 5.$$
The latter congruence would imply that $(2k+1)^2 \equiv -3 \pmod 5$; this would give in turn that
$$1 = \left(\frac{-3}{5}\right) = \left(\frac{-1}{5}\right)\left(\frac{3}{5}\right) = (-1)^{\frac{5-1}{2}}\left(\frac{5}{3}\right)(-1)^{\frac{5-1}{2}\cdot\frac{3-1}{2}} = \left(\frac{2}{3}\right) = (-1)^{\frac{3^2-1}{8}} = -1$$
which is decidedly absurd. It follows from all this that n cannot be equal to 10.

We are going to prove now that it is possible to find 11 consecutive integers whose squares add up to another perfect square. The sum of the squares of 11 consecutive integers is equal to
$$\sum_{j=1}^{11}(k+j)^2 = 11(k^2+12k+46)$$

for some $k \in \mathbb{Z}$. Thus, in order for $11(k^2 + 12k + 46)$ to be a perfect square, it is necessary that $k^2 + 12k + 46 \equiv 0 \pmod{11}$. Since $k^2 + 12k + 46 \equiv k^2 + k + 2 \pmod{11}$, the congruence $k^2 + 12k + 46 \equiv 0 \pmod{11}$ is equivalent to the congruence $k^2 + k + 2 \equiv 0 \pmod{11}$. The latter congruence is satisfied by the elements of exactly two congruence classes modulo 11: by the numbers congruent to 5 modulo 11 and by the numbers congruent to $11 - 5 = 6$ modulo 11. It follows that if $\sum_{j=1}^{11}(k + j)^2$ is a perfect square, then $k \in \{5, 16, 27, \ldots\}$ or $k \in \{6, 17, 28, \ldots\}$. After discarding the possibility that $k \in \{5, 6, 16\}$, we notice that when $k = 17$ we have that

$$\sum_{j=1}^{11}(k + j)^2 = \sum_{j=1}^{11}(17 + j)^2 = 11(17^2 + 12 \cdot 17 + 46) = 11 \cdot 539 = 7^2 \cdot 11^2$$

and we are done.

□

Scholium. It has to be noted that we have actually proved that $k = 7$ is the minimal natural number such that $\sum_{j=1}^{11}(k + j)^2$ is a perfect square.

S393 (Issue 6, 2016. Proposed by Titu Andreescu, University of Texas at Dallas, USA.). If n is an integer such that $n^2 + 11$ is a prime number, prove that $n + 4$ is not a perfect cube.

Solution. We establish the contrapositive proposition:
If $n + 4 = x^3$ for some $x \in \mathbb{Z}$, then

$$\begin{aligned} n^2 + 11 &= (x^3 - 4)^2 + 11 \\ &= x^6 - 8x^3 + 27. \end{aligned}$$

We claim that the latter trinomial can be factored by means of the well-known identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

Indeed, letting $a = x^2$, $b = x$, and $c = 3$, we get

$$\begin{aligned} x^6 - 8x^3 + 27 &= a^3 + b^3 + c^3 - 3abc \\ &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (x^2 + x + 3)(x^4 + x^2 + 9 - x^3 - 3x - 3x^2) \\ &= (x^2 + x + 3)(x^4 - x^3 - 2x^2 - 3x + 9). \end{aligned}$$

Since $x^6 - 8x^3 + 27 = x^3(x^3 - 8) + 27 > x^2 + x + 3 = \left(x + \frac{1}{2}\right)^2 + \frac{11}{4} > 1$, it follows that $x^2 + x + 3$ is a proper factor of $n^2 + 11$; therefore, $n^2 + 11$ is not a prime number.

□

U391 (Issue 6, 2016. Proposed by Alessandro Ventullo, Milan, Italy.). Find all positive integers n such that

$$\phi(n)^3 \leq n^2.$$

Solution. Our solution leans on the following lower estimate for the Euler totient function

$$\phi(n) > \left(\frac{23}{100}\right)\frac{n}{\log n}, \tag{22}$$

which is valid for every natural number $n > 1$. This estimate implies at once the finitude of the solution set of the inequality under consideration; it can be proven as follows:

If $n = p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ and $\sigma(n) = \sum_{d|n} d$, then on the one hand we have that

$$\begin{aligned}
\phi(n)\sigma(n) &= n\prod_{p|n}\left(1-\frac{1}{p}\right)\prod_{p|n}\left(\frac{p^{\alpha+1}-1}{p-1}\right)\\
&= n\prod_{p|n}\frac{p^{\alpha+1}-1}{p}\\
&= n\prod_{p|n}\frac{p^{\alpha+1}}{p}\left(1-\frac{1}{p^{\alpha+1}}\right)\\
&= n^2\prod_{p|n}\left(1-\frac{1}{p^{\alpha+1}}\right)\\
&\geq n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right)\\
&\geq n^2\prod_{p\in\mathbf{P}}\left(1-\frac{1}{p^2}\right)\\
&= \frac{n^2}{\zeta(2)}\\
&= \frac{6}{\pi^2}n^2. \qquad (23)
\end{aligned}$$

On the other hand,

$$\sigma(n) = \sum_{d|n}\frac{n}{d} \leq n\sum_{d=1}^{n}\frac{1}{d} \leq n(1+\log n) < 2.5\,n\log n. \tag{24}$$

From (23) and (24), we obtain

$$\phi(n) > \frac{6n^2}{\pi^2(2.5\,n\log n)} > \left(\frac{23}{100}\right)\frac{n}{\log n}$$

which was what we wanted.

Having (22) at our disposal, we proceed to show that the inequality $\phi(n)^3 \leq n^2$ cannot hold if $n > e^{12}$. Let us consider the function $f\colon (1,\infty) \to \mathbb{R}$ defined by $f(n) = \left(\frac{100}{23}\right)^3\log^3 n - n$ for every $n \in (1,\infty)$. Resorting to Mathematica or some other computer algebra system, we notice that $f(e^{12}) \approx -20\,731.3$; furthermore, since $f'(n) = 3\left(\frac{100}{23}\right)^3\frac{\log^2 n}{n} - 1$ and $f''(n) = 3\left(\frac{100}{23}\right)^3\frac{2\log n - \log^2 n}{n^2}$, we infer that $f'(n)$ is strictly decreasing in the interval (e^2,∞). From this and the fact that $f'(e^{11}) \approx -0.501708$, we get that $f'(n) < 0$ for every $n \in I := (e^{12},\infty)$: this implies in particular that f is strictly decreasing in I and, what is more, that $f(n) < 0$ for every $n \in I$. Thus, if $n > e^{12}$ then $\left(\frac{100}{23}\right)^3 \log^3 n < n$ and whence

$$n^2 < \left(\frac{23}{100}\right)^3 \frac{n^3}{\log^3 n} = \left(\left(\frac{23}{100}\right)\frac{n}{\log n}\right)^3 < \phi(n)^3.$$

Hence, in order to finish up the solution we only need to determine which $n \in \mathbb{N} \cap [1, \lfloor e^{12} \rfloor]$ satisfy the inequality $\phi(n)^3 \leq n^2$. This can be done in Mathematica via the following loop:

```
For[n=1, n<=Floor[E^12], n++, If[(EulerPhi[n])^3<=n^2, Print[n]]]
```

The output we get allows us to conclude that $\phi(n)^3 \leq n^2$ iff $n \in \{1, 2, 3, 4, 6, 8, 10, 12, 18, 24, 30, 42\}$.

□

J397 (Issue 1, 2017. Proposed by Adrian Andreescu, Dallas, Texas, USA.) Find all positive integers n for which $3^4 + 3^5 + 3^6 + 3^7 + 3^n$ is a perfect square.

Solution. Since $3^4 + 3^5 + 3^6 + 3^7 = 3^4(1 + 3 + 3^2 + 3^3) = 3^4(40) \equiv 0 \pmod 4$, it follows that n must be an even natural number.

Let us denote the sum $3^4+3^5+3^6+3^7+3^n = 3^4(40+3^{n-4})$ by a_n. Since $a_2 = 3^2(3^2\cdot 40+1) = 3^2\cdot 19^2$, we wish to determine all the even natural numbers $n \geq 4$ such that $40 + 3^{n-4} = m^2$ for some (odd) natural number m. This equation is equivalent to

$$40 = (m - 3^{\frac{n-4}{2}})(m + 3^{\frac{n-4}{2}}).$$

Since the first factor in the right-hand side of the equation is less than the second one and both of them are even, we infer that either $m - 3^{\frac{n-4}{2}} = 2$ or $m - 3^{\frac{n-4}{2}} = 4$. In the former case we obtain that $m = 11$ and $n = 8$ while in the latter we get that $m = 7$ and $n = 6$.

Then, given that $a_6 = 3^4(40 + 3^2) = 3^4 \cdot 7^2$ and $a_8 = 3^4(40 + 3^4) = 3^4 \cdot 11^2$, we conclude that $a_n := 3^4 + 3^5 + 3^6 + 3^7 + 3^n$ is perfect square if and only if $n \in \{2, 6, 8\}$.

□

J401 (Issue 1, 2017. Proposed by Adrian Andreescu, Dallas, Texas, USA.) Find all integers n for which $2^n + n^2$ is a perfect square.

Solution. If $n = 2k + 1$ for some $k \in \mathbb{Z}^+$ and $2^n + n^2$ is a perfect square, then

$$2^n = (2u + 1)^2 - (2k + 1)^2 = 4(u - k)(u + k + 1)$$

for some $u \in \mathbb{N}$. Assuming that $n \geq 3$, we rewrite this equality as

$$2^{n-2} = (u - k)(u + k + 1).$$

Since 2 cannot be a common factor of $u-k$ and $u+k+1$, we infer that $u-k = 1$, $u+k+1 = 2^{n-2}$, and whence

$$n = 2k + 1 = 2^{n-2} - 1.$$

From this and the inequality $2^{n-2} - 1 > n$, which is valid for every natural number $n \geq 5$, we obtain that there is no odd natural number n such that $2^n + n^2$ is a perfect square.

If $n = 2k$ for some natural number $k \geq 7$, then $2^n + n^2 = 2^{2k} + (2k)^2$ is not a perfect square because it lies strictly between two consecutive perfect squares:

$$(2^k)^2 < 2^{2k} + (2k)^2 < (2^k + 1)^2.$$

Combining the results in the previous two paragraphs and considering the computations registered in the table below, we conclude that $2^n + n^2$ is a perfect square if and only if $n = 0$ or $n = 6$.

n	$2^n + n^2$
0	1
2	8
4	32
6	100
8	320
10	1124
12	4240

□

S400 (Issue 1, 2017. Proposed by Titu Andreescu, Dallas, Texas, USA.) Find all n for which $(n-4)! + \frac{1}{36n}(n+3)!$ is a perfect square.

Solution. We have that

$$\begin{aligned}
(n-4)! + \frac{1}{36n}(n+3)! &= (n-4)!\left(\frac{36+(n-3)(n-2)(n-1)(n+1)(n+2)(n+3)}{36}\right)\\
&= (n-4)!\left(\frac{36+(n^2-9)(n^2-4)(n^2-1)}{36}\right)\\
&= \frac{(n-4)!(n^6-14n^4+49n^2)}{36}\\
&= (n-4)!\left(\frac{n(n^2-7)}{6}\right)^2.
\end{aligned}$$

Since $6 \mid n(n^2-7)$ for every $n \in \mathbb{N}$, it follows that $(n-4)! + \frac{1}{36n}(n+3)!$ is perfect square if and only if $(n-4)!$ is a perfect square. By resorting to Bertrand's Postulate it can easily be shown that $N!$ is a perfect square if and only if $N = 0$ or $N = 1$ (cf. the solution to problem O100 or J191). Hence, $(n-4)! + \frac{1}{36n}(n+3)!$ is a perfect square if and only if $n \in \{4, 5\}$.

□

O397 (Issue 1, 2017. Proposed by Titu Andreescu, Dallas, Texas, USA.) Solve in integers the equation:

$$(x^3-1)(y^3-1) = 3(x^2y^2+2).$$

Solution. The equation is equivalent to

$$x^3y^3 - x^3 - y^3 - 3x^2y^2 = 5. \tag{25}$$

Then, applying the well-known identity $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ to the expression in the left-hand side of (25), we can further rewrite the original equation as

$$(xy-x-y)((xy)^2+x^2+y^2+x^2y-xy+xy^2) = 5.$$

Since 5 is a prime number, it follows that $xy-x-y=\pm1$ or $xy-x-y=\pm5$:

If $xy-x-y=1$, then $(x-1)(y-1)=2$. There are four possibilities for the pair (x,y) in this case: $(x=2, y=3), (x=0, y=-1), (x=3, y=2)$, and $(x=-1, y=0)$. None of these pairs is a solution of the original equation.

If $xy-x-y=-1$, then $(x-1)(y-1)=0$. Since $3(x^2y^2+2)>0$, no solution of $(x-1)(y-1)=0$ yields a solution of the original equation.

If $xy-x-y=5$, then $(x-1)(y-1)=6$. There are eight possibilities for the pair (x,y) in this case: $(x=2, y=7), (x=0, y=-5), (x=3, y=4), (x=-1, y=-2), (x=4, y=3), (x=-2, y=-1), (x=7, y=2)$, and $(x=-5, y=0)$. Within these pairs, there are only two which satisfy the original equation: $(x=-1, y=-2)$ and $(x=-2, y=-1)$.

If $xy-x-y=-5$, then $(x-1)(y-1)=-4$. There are six possibilities for the pair (x,y) in this case: $(x=2, y=-3), (x=3, y=-1), (x=5, y=0), (x=0, y=5), (x=-1, y=3)$, and $(x=-3, y=2)$. We see that none of these six pairs yields a solution of the original equation, either.

The previous analysis allows us to conclude that there are only two solutions to the given equation: $(x=-1, y=-2)$ and $(x=-2, y=-1)$.

□

S430 (Issue 6, 2017. Proposed by Florin Rotaru, Focşani, Romania.) Prove that

$$\sin\frac{\pi}{2n} \geq \frac{1}{n}$$

for all positive integers n.

Solution. The given inequality is a special instance of a slightly more general one, namely

$$\sin(x) \geq \frac{2}{\pi}x \quad \text{for every} \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{26}$$

This inequality can be proven as follows. Let us consider the function $f\colon [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by $f(u) = -\sin(u)$ for every $u \in [0, \frac{\pi}{2}]$. Since $f''(u) = \sin(u) \geq 0$ for every $u \in [0, \frac{\pi}{2}]$, it follows that f is a convex function. Hence, for every $t \in [0,1]$ and $(v, w) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ where $v < w$, the inequality

$$f(tw + (1-t)v) \leq tf(w) + (1-t)f(v) \tag{27}$$

holds true. In particular, for any given $x \in [0, \frac{\pi}{2}]$, by letting $t := \frac{2}{\pi}x$, $v := 0$, and $w := \frac{\pi}{2}$ in (27) we obtain that

$$\begin{aligned} f(x) &\leq \frac{2}{\pi}xf\left(\frac{\pi}{2}\right), \\ -\sin(x) &\leq -\frac{2}{\pi}x \end{aligned}$$

and the proof terminates.

□

Note. Since the equation of the line in $\mathbb{R}^2$ that passes through the points $(0, \sin(0))$ and $(\pi/2, \sin(\pi/2))$ is $y = \frac{2}{\pi}x$, the inequality (26) does beg for a proof via convexity.

U438 (Issue 1, 2018, Proposed by José Hernández Santiago, Michoacán, México.) Prove that a natural number n is of the form $x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ iff $v_2(n) \neq 1$ and, for every prime number $p \equiv 3, 5, 6 \pmod 7$, $v_p(n)$ is even.

First solution. Our first proof of this proposition leans on the following four preliminary results:

Lemma. The product of two numbers of the form $x^2 + 7y^2$ is equal to another number of the same form.

Proof. If $a, b, c, d \in \mathbb{R}$, then

$$(a + i\sqrt{7}b)(c + i\sqrt{7}d) = (ac - 7bd) + i\sqrt{7}(ad + bc).$$

Therefore,

$$\begin{aligned}(ac - 7bd)^2 + 7(ad + bc)^2 &= |(ac - 7bd) + i\sqrt{7}(ad + bc)|^2 \\ &= |(a + i\sqrt{7}b)(c + i\sqrt{7}d)|^2 \\ &= |a + i\sqrt{7}b|^2|c + i\sqrt{7}d|^2 \\ &= (a^2 + 7b^2)(c^2 + 7d^2).\end{aligned}$$

□

Theorem. Let p be an odd prime number and D a natural number such that $p \nmid D$ and $\left(\frac{-D}{p}\right) = 1$. Then, there exists $(k, x, y) \in \mathbb{Z}^3$ such that $0 < k \le D$, $0 < x, y < \sqrt{p}$, and $x^2 + Dy^2 = kp$.

Proof. (d'après A. Thue) Let s be an integer which satisfies the congruence $s^2 \equiv -D \pmod p$ and let us denote the set $\{0, 1, \ldots, \lfloor\sqrt{p}\rfloor\}$ by $\mathcal{S}$. Then, a simple cardinality argument allows us to ascertain the existence of two elements of the set $\{t - su : (t, u) \in \mathcal{S} \times \mathcal{S}\}$ that are congruent modulo p. Let us suppose that these two elements correspond to the two (distinct) pairs $(t, u), (v, w) \in \mathcal{S} \times \mathcal{S}$. Then, if $x := |t - v|$ and $y := |u - w|$, it follows that $(x, y) \in \mathcal{S} \times \mathcal{S}$ and $x \equiv \pm sy \pmod p$. Since x, y are not simultaneously equal to zero we have that

$$0 < x^2 + Dy^2 < p + Dp = (1 + D)p. \tag{28}$$

On the other hand, it is the case that

$$x^2 + Dy^2 \equiv s^2y^2 + Dy^2 \equiv 0 \pmod p. \tag{29}$$

It follows from (28) and (29) that $x^2 + Dy^2 = kp$ for some $k \in (0, D] \cap \mathbb{N}$ and we are done. □

Corollary 1. Let $p \neq 2, 7$ be a prime number. Then, $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ iff $\left(\frac{-7}{p}\right) = 1$.

Proof. [⇒] If $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$, then $p \nmid y$. Then, the thesis in question follows in this case after multiplying both sides of the congruence $x^2 \equiv -7y^2 \pmod p$ by the square of an integer z such that $yz \equiv 1 \pmod p$.

[⇐] If $p \neq 2, 7$ and $\left(\frac{-7}{p}\right) = 1$, then the theorem above guarantees the existence of $(k, x, y) \in \mathbb{Z}^3$ such that $0 < k \le 7$, $0 < x, y < \sqrt{p}$, and $x^2 + 7y^2 = kp$. We claim that from $x^2 + 7y^2 = kp$ and the condition $0 < k \le 7$, it can be deduced that $p = X^2 + 7Y^2$ for some $X, Y \in \mathbb{Z}$:

The case $k = 2$ is impossible, otherwise the left-hand side of $x^2 + 7y^2 = 2p$ would be divisible by 4. The case $k = 3$ is also impossible, otherwise the left-hand side of $x^2 + 7y^2 = 3p$ would be a multiple of 9: this would imply in turn that $p = 3$, which is absurd because $\left(\frac{-7}{3}\right) \neq 1$. If $k = 4$ and $x^2 + 7y^2 = 4p$, then x, y are integers of the same parity. If both of them are odd, then the left-hand side of $x^2 + 7y^2 = 4p$ is divisible by 8 (which is absurd given that $p \neq 2$). If $x = 2u$ and $y = 2v$ for some $u, v \in \mathbb{Z}$, then $p = \frac{x^2+7y^2}{4} = \frac{4u^2+7(4v^2)}{4} = u^2 + 7v^2$. In the case $k = 5$, a necessary condition for the equality $x^2 + 7y^2 = 5p$ to hold is that $x = 5u$ and $y = 5v$ for some $u, v \in \mathbb{Z}$; since this would imply that $5 \mid p$, this case can be discarded too (-7 is not a quadratic residue modulo 5). Finally, in the case $k = 6$, a necessary condition for the equality $x^2 + 7y^2 = 6p$ to hold is that $x = 3u$ and $y = 3v$ for some $u, v \in \mathbb{Z}$; since this would imply that $3 \mid p$, this case can also be discarded (-7 is not a quadratic residue modulo 3). □

Corollary 2. Let p be an odd prime number. Then, $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ iff $p = 7$ or $p \equiv 1, 2, 4 \pmod 7$.

Proof. It is a straightforward consequence of the previous corollary. □

We are ready to provide a solution to the problem now:

[⇒] If $n = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ and $2 \mid n$, then x and y are integers of the same parity: if both of them are even, $v_2(n) \geq 2$; if both of them are odd, $v_2(n) \geq 3$. Now, let us suppose that p is a prime divisor of n which is congruent to 3, 5 or 6 modulo 7. In order to reach a contradiction, let us assume that $v_p(n)$ is odd. If $d = \gcd(x, y)$, then $n = d^2(x_0^2 + 7y_0^2)$ for some coprime integers x_0, y_0. From this equality and the assumption we made on $v_p(n)$, it follows that $p \mid x_0^2 + 7y_0^2$ which clearly conflicts with the fact that -7 is not a quadratic residue modulo p.

[⇐] For every $\alpha \in \mathbb{Z}^+ \setminus \{1\}$, 2^α is a number of the form $x^2 + 7y^2$. The required conclusion follows from corollary 2 and the lemma.

□

Second solution. One of the main differences between this proof and the former is that one of the preliminary lemmas is established now via Minkowski's first theorem in the geometry of numbers (cf. M. B. Nathanson, *Additive number theory (Inverse problems and the geometry of sumsets)*. Springer-Verlag, 1996, pp. 167-177.).

Lemma 1. The product of two numbers of the form $x^2 + 7y^2$, where both x and y belong to $\mathbb{R}$, is equal to another number of the same form.

Proof. If $a, b, c, d \in \mathbb{R}$, then

$$(a + i\sqrt{7}b)(c + i\sqrt{7}d) = (ac - 7bd) + i\sqrt{7}(ad + bc).$$

Therefore,

$$\begin{aligned}(ac - 7bd)^2 + 7(ad + bc)^2 &= |(ac - 7bd) + i\sqrt{7}(ad + bc)|^2 \\ &= |(a + i\sqrt{7}b)(c + i\sqrt{7}d)|^2 \\ &= |a + i\sqrt{7}b|^2|c + i\sqrt{7}d|^2 \\ &= (a^2 + 7b^2)(c^2 + 7d^2).\end{aligned}$$

□

Lemma 2. Let p be an odd prime number. If $p \equiv 1, 2, 4 \pmod 7$, then $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$.

Proof. If p is an odd prime number in any of those equivalence classes modulo 7, then we can guarantee the existence of $s \in \mathbb{Z}$ such that $s^2 \equiv -7 \pmod p$. Let us consider the centrally symmetric convex body $\mathcal{D} := \{(x, y) \in \mathbb{R}^2 : x^2 + 7y^2 < 4p\}$ and the lattice $\Lambda := \{(x, y) \in \mathbb{Z}^2 : x \equiv sy \pmod p\} = \langle (p, 0), (s, 1) \rangle$. Since

$$\operatorname{vol}(\mathcal{D}) = \pi\sqrt{4p}\sqrt{4p/7} = \frac{4\pi}{\sqrt{7}}p, \qquad \det(\Lambda) = \left|\det\begin{pmatrix} p & s \\ 0 & 1 \end{pmatrix}\right| = p,$$

and $\operatorname{vol}(\mathcal{D}) > 2^2 \det(\Lambda)$, Minkowski's first theorem allows us to ascertain the existence of $(x_0, y_0) \in (\mathcal{D} \cap \Lambda) \setminus \{(0, 0)\}$. Then, given that $(x_0, y_0) \in \Lambda$, it follows that $x_0^2 \equiv s^2 y_0^2 \pmod 7$ and whence $7 \mid x_0^2 + 7y_0^2$. On the other hand, since (x_0, y_0) is a non-zero element of $\mathcal{D}$, we have that $0 < x_0 + 7y_0^2 < 4p$. Thus, $x_0^2 + 7y_0^2 = p$ or $x_0^2 + 7y_0^2 = 2p$ or $x_0^2 + 7y_0^2 = 3p$. The second case is impossible because $2 \mid x_0^2 + 7y_0^2$ necessarily implies that $4 \mid x_0^2 + 7y_0^2$ (whereas $2 \parallel 2p$); the third case is impossible because $3 \mid x_0^2+7y_0^2$ necessarily implies $9 \mid x_0^2+7y_0^2$ (whereas $3 \parallel 3p$). Therefore, $p = x_0^2 + 7y_0^2$ and we are done. □

We are ready to provide a solution to the problem now:

[⇒] If $n = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ and $2 \mid n$, then x and y are integers of the same parity: if both of them are even, $v_2(n) \geq 2$; if both of them are odd, $v_2(n) \geq 3$. If $n = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$ and $2 \nmid n$, then $v_2(n) = 0$. Now, let us suppose that p is a prime divisor of n which is congruent to 3, 5 or 6 modulo 7. In order to reach a contradiction, let us assume that $v_p(n)$ is odd. If $d = \gcd(x, y)$, then $n = d^2(x_0^2 + 7y_0^2)$ for some coprime integers x_0, y_0. From this equality and the assumption we made on $v_p(n)$, it follows that $p \mid x_0^2 + 7y_0^2$ which clearly conflicts with the fact that -7 is not a quadratic residue modulo p.

[⇐] Let us suppose that $n = 2^\alpha \cdot p_1^{\beta_1} \cdots p_r^{\beta_r} \cdot q_1^{\gamma_1} \cdots q_s^{\gamma_s} \cdot 7^\omega$ where the primes p_i are congruent to 3, 5 or 6 modulo 7 and the primes q_j are all odd and congruent to $1, 2$ or 4 modulo 7. Then, the desired conclusion is an straightforward consequence of the assumption that the exponents $\beta_1, \ldots, \beta_r$ are even, the fact that for every $\alpha \in \mathbb{Z}^+ \setminus \{1\}$ (resp. $\omega \in \mathbb{Z}^+$), 2^α (resp. 7^ω) is a number of the form $x^2 + 7y^2$, and lemmas 1 and 2. □