Solving problems by looking at a discriminant

José Hernández Santiago

1 Introduction

The discriminant of the quadratic polynomial $p(x) = ax^2 + bx + c$ is equal to the quantity $b^2 - 4ac$. It is well-known that this discriminant provides information regarding the zeros of any $p(x)$ having real coefficients without calculating them explicitly. An example of a prominent result that can be proven by exploiting this sort of information is the Cauchy-Schwarz inequality (see, for instance, [7, p. 11]).

Our purpose in this paper is **to illustrate** the importance of taking discriminants of quadratic polynomials into account while solving problems that have to do with the resolution of a Diophantine equation in two or more variables. To be more precise: the type of problem with which we are going to deal herein reduces to determining the k-tuples $(x_1, x_2, \ldots, x_k)$ in $\mathbb{Z}^k$ (or in $\mathbb{N}^k$) satisfying

$$F(x_1, x_2, \ldots, x_k) = 0 \tag{1}$$

where F is a k-ary function with $k \geq 2$; the basic idea to which we are going to resort in order to determine those k-tuples $(x_1, x_2, \ldots, x_k)$ consists in considering an auxiliary quadratic polynomial in one indeterminate $p(x)$ and deriving necessary conditions for the equality in (1) to hold from the analysis of the discriminant of $p(x)$. Clearly enough, the quadratic polynomial $p(x)$ is going to be related to the function $F(X_1, X_2, \ldots, X_k)$; when looking for it, the main hint is to pay close attention to any of the variables $X_1, X_2, \ldots, X_n$ with respect to which F is a quadratic polynomial.

Since the coefficients of the auxiliary quadratic polynomial are going to be integers, it is principally through the following lemma about discriminants that we will obtain necessary conditions for the solvability (or non-solvability) of any equation resembling (1) that we treat in our paper:

> MAIN LEMMA. If the coefficients of the polynomial $p(x) = ax^2 + bx + c$ are integers and one of the zeros of $p(x)$ belongs to $\mathbb{Z}$, then the discriminant of $p(x)$ is a perfect square.

Innocuous though this fact may seem, we aim to show in what follows how it is that– by having it in mind and appealing to it whenever it is possible to do so–we can settle a host of olympiad-caliber problems in a methodical way. Naturally, the set of examples with which we will be working is not exhaustive; further situations wherein this idea of analyzing a discriminant is of some aid can be found in [1, pp. 25-26, 196-197, and 274].

2 First examples

Let us provide a detailed example of a problem that can be tackled by considering an auxiliary quadratic polynomial and analyzing its discriminant via the main lemma. This problem can be found in [6, p. 27].

Example 1. *Let a and b natural numbers such that $2a^2 + a = 3b^2 + b$. Prove that $a - b$ is a perfect square.*

Solution. The equality $2a^2 + a = 3b^2 + b$ is equivalent to

$$a^2 + 3(b^2 - a^2) - (a - b) = 0$$

or to

$$a^2 + 6(b-a)a + [3(b-a)^2 - (a-b)] = 0.$$

The existence of $a, b \in \mathbb{N}$ satisfying any of these equalities implies that the roots of the quadratic equation with integer coefficients

$$\mathcal{X}^2 + 6(b-a)\mathcal{X} + [3(b-a)^2 - (a-b)] = 0 \tag{2}$$

belong to $\mathbb{Z}$ (in point of fact, one of the roots is $x_1 = a$ and the other is $x_2 = 5a - 6b$). It follows from the main lemma that the discriminant Δ of the quadratic polynomial in the left-hand side of (2) is a positive perfect square. Since

$$\begin{aligned} \Delta &= 36(b-a)^2 - 4[3(b-a)^2 - (a-b)] \\ &= 24(a-b)^2 + 4(a-b) \\ &= 4(a-b)[6(a-b)+1] \end{aligned}$$

and $\Delta = c^2$ for a certain $c \in \mathbb{N}$, we get that

$$4(a-b)[6(a-b)+1] = c^2.$$

From this equality and the coprimality of $(a-b)$ and $6(a-b)+1$ we conclude that both $a-b$ and $6(a-b)+1$ are perfect squares and our proof terminates. □

For the sake of a better understanding of the last part of our solution, we are going to formulate explicitly the result that allowed us to infer that both $a-b$ and the other number are perfect squares; it is a consequence of the fundamental theorem arithmetic whose proof we leave as an exercise for the reader.

Theorem 1. *Let $a, b, c \in \mathbb{N}$. If $(a, b) = 1$ and $a \cdot b = c^2$, then both a and b are perfect squares.*

Even though the solution we have offered for this problem is slightly longer than the solution that was put forward in [6, p. 152], by considering the discriminant of the auxiliary polynomial we also derived the "squareness" of $6a - 6b + 1$ (a conclusion that is not exactly immediate to foresee from the mere formulation of the problem). As we mentioned above, we wish to illustrate in our paper that the idea of considering an auxiliary quadratic polynomial and deriving additional information from the analysis of its discriminant may be applied in many situations.

Example 2. *(Iberoamerican Mathematical Olympiad; pb. 5, 2015.) Determine all pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $(b^2 + 7(a-b))^2 = a^3 b$.*

Solution. Taking the square of the binomial in the left-hand side and substracting a^3b from both sides, we see that the given equation is equivalent to

$$b(b^3 - a^3) + 14b^2(a-b) + 49(a-b)^2 = 0.$$

This equation can be rewritten in turn as

$$(a-b)[-ba^2 + (49 - b^2)a - b(b-7)^2] = 0.$$

Hence, if $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, with $a \neq b$, is a solution to $(b^2 + 7(a-b))^2 = a^3b$ then

$$-ba^2 + (49 - b^2)a - b(b-7)^2 = 0.$$

Given that the expression in the left-hand side is of second degree in a, the auxiliary equation we are going to consider in this problem is

$$-b\mathcal{X}^2 + (49 - b^2)\mathcal{X} - b(b-7)^2 = 0. \tag{3}$$

Since the coefficients of the polynomial in the left-hand side of the above equation belong all to $\mathbb{Z}$ and one of its roots is $x_1 = a$, it follows that its discriminant Δ_b is a perfect square. Now then, from

$$\begin{aligned}\Delta_b &= (49 - b^2)^2 - 4b^2(b-7)^2 \\ &= (7-b)^2(7+b)^2 - 4b^2(b-7)^2 \\ &= (7-b)^2(7-b)(7+3b),\end{aligned}$$

we get that b is of necessity an integer in the interval $[-2, 7]$. By considering the prime decomposition of the product $(7-b)(7+3b)$ for each $b \in [-2, 7]$, we infer that Δ_b may be a perfect square only when $b \in B = \{-2, 0, 3, 6, 7\}$. We proceed to calculate the solutions of (3) for every $b \in B \setminus \{0\}$. The corresponding results can be found in the table below.

b	Δ_b	$\dfrac{(b^2-49)+\sqrt{\Delta_b}}{-2b}$	$\dfrac{(b^2-49)-\sqrt{\Delta_b}}{-2b}$
-2	$9^2(9)$	$-\frac{18}{4}$	-18
3	$4^2(4)(16)$	$\frac{8}{6}$	12
6	25	$\frac{8}{12}$	$\frac{18}{12}$
7	0	0	0

We conclude from this analysis that the set of solutions to the Diophantine equation $(b^2+7(a-b))^2 = a^3b$ is $\{(z,z)\colon z \in \mathbb{Z}\} \cup \{(0,7), (12,3), (-18,-2)\}$. $\square$

A solution to this problem involving casework along different lines can be found in [8, pp. 66-68].

The reader can test his/her comprehension of what we have been discussing so far with the following problem from the 2014 Hong Kong Mathematical Olympiad; this problem and the previous one are of the same kind but the one from Hong Kong is a bit easier to solve**.

Exercise 1. *Find all pairs* $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ *satisfying* $(b^2 + 11(a-b))^2 = a^3b$.

3 A second round of examples

The next example first appeared in the section "Problems contributed by the readers" of the August 1984 issue of the Russian journal *Kvant*. It was contributed by M. Z. Garaev, a ninth-grade student back then.

Example 3. *Determine whether or not the equation* $x^3 + y^3 = (x+y)^2 + (xy)^2$ *has a solution in natural numbers* x *and* y.

Solution. Factoring the sum of cubes in the left-hand of the equation, we may rewrite it as follows

$$(x+y)((x+y)^2 - 3xy) = (x+y)^2 + (xy)^2.$$

Then, considering the change of variables $u = x + y$ and $v = xy$, the original equation becomes

$$u(u^2 - 3v) = u^2 + v^2$$

or equivalently

$$v^2 + 3uv + (u^2 - u^3) = 0. \tag{4}$$

**If we understand correctly, in one of the exams of the 2014 Belarusian Mathematical Olympiad there appeared a problem that might be a common ancestor of these two problems. Since we have not been able to find its exact formulation on the internet, we would greatly appreciate any information regarding it that the readers may wish to share with us.

This is a second degree equation in the variable v; if it is to admit solutions in natural numbers, then the discriminant

$$\Delta_1 = 9u^2 - 4(u^2 - u^3) = 5u^2 + 4u^3 = u^2(4u + 5)$$

has to be a positive perfect square. It follows that $4u + 5 = (2k + 1)^2$ for a natural number $k > 1$; this fact and (4) imply in turn that

$$u = k^2 + k - 1 \qquad \text{and} \qquad v = \frac{-3u + \sqrt{\Delta_1}}{2} = \frac{-3u + u(2k + 1)}{2} = u(k - 1) = (k^2 + k - 1)(k - 1).$$

Further, since $v = xy = (u - y)y$, the existence of solutions to the original equation implies that discriminant Δ_2 of the (second) auxiliary quadratic polynomial

$$\mathcal{Y}^2 - u\mathcal{Y} + v \quad = \quad \mathcal{Y}^2 - (k^2 + k - 1)\mathcal{Y} + (k^2 + k - 1)(k - 1)$$

is a perfect square. Provided that

$$\Delta_2 = (k^2 + k - 1)^2 - 4(k^2 + k - 1)(k - 1) = (k^2 + k - 1)[(k^2 + k - 1) - 4(k - 1)]$$

and

$$(k^2 + k - 1 - 4(k - 1), k^2 + k - 1) = 1,$$

Theorem 1 yields that $k^2 + k - 1$ is a perfect square; this is, however, a conclusion that for $k > 1$ can't be true because, in that range for k, the number $k^2 + k - 1$ lies strictly between two consecutive perfect squares:

$$k^2 < k^2 + k - 1 < (k + 1)^2.$$

Therefore, there are no natural numbers x and y such that $x^3 + y^3 = (x + y)^2 + (xy)^2$. $\square$

There are two features that distinguish this example from those we had previously presented. In the first place, since there was not a suitable quadratic relationship between the original variables, we had to make a change of variables before we thought of introducing the first auxiliary quadratic polynomial. Secondly, to reach our conclusion we didn't resort to one but to two auxiliary quadratic polynomials. It may be worthwhile to flesh out the result we used in order to assure that, for a natural number $k > 1$, the number $k^2 + k - 1$ is not a perfect square:

Theorem 2. *If c_0 is a natural number and $n^2 < c_0 < (n + 1)^2$ for a natural number n, then c_0 can't be a perfect square.*

This theorem is a (straightforward) consequence of the fact that there are no integers in the interval $(0, 1)$. When facing a problem related to perfect squares, it is a tool to have in mind; it has cropped up in the pages of *Mathematical Reflections* in several occasions in the past (cf. solutions to problems J111 (issue #1, 2009), J159 (issue #3, 2010), S265 (issue #3, 2013), J401 (issue #1, 2017), J503 (issue #6, 2019), ...).

On to another example:

Example 4. *(USAMO; pb. 1, 1987.) Find all solutions to $(m^2 + n)(m + n^2) = (m - n)^3$, where m and n are non-zero integers.*

Solution. The equation can be rewritten as

$$m^2(n^2 + 3n) + m(n - 3n^2) + 2n^3 = 0. \tag{5}$$

It is clear that this equation does not have any solutions when $n = -3$. For any given $n \neq -3$, we can regard it as a second degree equation in the variable m: if it admits integer solutions as an equation in m, then its discriminant Δ must be a perfect square. Since

$$\Delta = (n - 3n^2)^2 - 4(n^2 + 3n)(2n^3) = n^2(1 - 6n - 15n^2 - 8n^3)$$

and the expression inside the parentheses is negative for every $n \in \mathbb{N}$, we are going to suppose in what follows that $n = -N$ for some $N \in \mathbb{N} \setminus \{3\}$. The discriminant becomes then

$$\Delta = N^2(8N^3 - 15N^2 + 6N + 1) = N^2(N-1)^2(8N+1);$$

given that we need it to be a perfect square, we must have

$$N = \frac{a(a+1)}{2}$$

for an $a \in \mathbb{N} \setminus \{2\}$. The idea is to use this information and the equation (5) to derive a condition restricting the values a can take. Applying the quadratic formula in (5), we get that

$$m = \frac{-(n-3n^2) \pm \sqrt{\Delta}}{2(n^2+3n)} = \frac{(3N^2+N) \pm N(N-1)\sqrt{8N+1}}{2(N^2-3N)} = \frac{(3N+1) \pm (N-1)\sqrt{8N+1}}{2(N-3)}.$$

Each choice of the sign in the numerator yields information about a; we analyze the two possibilities separately.

If we choose the + sign, then

$$\begin{aligned} m &= \frac{(3N+1) + (N-1)\sqrt{8N+1}}{2(N-3)} \\ &= \frac{\left(\frac{3a(a+1)+2}{2}\right) + \left(\frac{a(a+1)-2}{2}\right)(2a+1)}{2\left(\frac{a(a+1)-6}{2}\right)} \\ &= \frac{(3a(a+1)+2) + (a(a+1)-2)(2a+1)}{2(a(a+1)-6)} \\ &= (a+2) + \frac{4}{a-2}. \end{aligned}$$

It follows that $a \in \{1, 3, 4, 6\}$. Each of these possibilities for a yields a solution (m, n) to the original equation:

a	$m = (a+2) + \frac{4}{a-2}$	$n = -\frac{a(a+1)}{2}$
1	-1	-1
3	9	-6
4	8	-10
6	9	-21

If we choose the − sign, then

$$\begin{aligned} m &= \frac{(3N+1) - (N-1)\sqrt{8N+1}}{2(N-3)} \\ &= \frac{\left(\frac{3a(a+1)+2}{2}\right) - \left(\frac{a(a+1)-2}{2}\right)(2a+1)}{2\left(\frac{a(a+1)-6}{2}\right)} \\ &= \frac{(3a(a+1)+2) - (a(a+1)-2)(2a+1)}{2(a(a+1)-6)} \\ &= (1-a) - \frac{4}{a+3}. \end{aligned}$$

In this case, the only admissible value for a is 1 and the solution for the original equation that it gives is $(m, n) = (-1, -1)$.

Thus, there are only four pairs (m, n) of non-zero integers that satisfy the equation under consideration: $(-1, -1)$, $(9, -6)$, $(8, -10)$, and $(9, -21)$. $\square$

The next problem that we are going to discuss appeared in the September 2015 issue of *Math Horizons* (specifically, in the pages of its section "The Playground"). However, it was also the subject matter of J. H. Jaroma's paper [3].

Example 5. *Determine all the natural numbers* n *such that the decimal representation of* $1 + 2 + \cdots + n$ *consists only of* 1*'s.*

Solution. Before we proceed to solve it, we observe that the problem is about triangular numbers and repunits. The n-th triangular number T_n is equal to $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$; for its part, the k-th repunit is the number whose decimal representation consists of exactly k 1's. The k-th repunit is usually denoted by R_k and the first repunits are

$$R_1 = 1, \quad R_2 = 11, \quad R_3 = 111, \quad R_4 = 1\,111, \quad R_5 = 11\,111, \ldots$$

Thus, the problem is asking to determine all the natural numbers that belong to both the sequence of repunits and the sequence of triangular numbers.

Since $R_k = \frac{10^k-1}{9}$ for every $k \in \mathbb{N}$, we need to find all the pairs $(n, k) \in \mathbb{N} \times \mathbb{N}$ satisfying the equation

$$\frac{n(n+1)}{2} = \frac{10^k - 1}{9}. \tag{6}$$

This equation is equivalent to

$$9n^2 + 9n - 2(10^k - 1) = 0,$$

which is of second degree in n and which we will take as the auxiliary equation for this problem. Provided that we are interested in finding the solutions to (6) in natural numbers n and k, the discriminant Δ of the polynomial $9n^2 + 9n - 2(10^k - 1)$ has to be a perfect square. In view of the fact that

$$\Delta = 9^2 - 4(9)(-2)(10^k - 1) = 81 + 72(10^k - 1) = 9(8 \cdot 10^k + 1),$$

it must be the case that

$$8 \cdot 10^k + 1 = x^2 \tag{7}$$

for a certain $x \in \mathbb{N}$.

For any $k \in \mathbb{N}$, the number in the left-hand side of the previous equality is congruent to 1 modulo 10. Consequently, $x = 10u \pm 1$ for some $u \in \mathbb{N}$ and the equation in (7) becomes

$$8 \cdot 10^k = 10u(10u \pm 2)$$

or equivalently

$$2 \cdot 10^k = 5u(5u \pm 1).$$

From this equality and the coprimality of $5u$ and $5u \pm 1$, we obtain that $5u = 5^k$ and $5u \pm 1 = 2^{k+1}$. Therefore, it all basically boils down to finding all the natural numbers k such that

$$2^{k+1} - 5^k = 1 \tag{8}$$

or

$$5^k - 2^{k+1} = 1. \tag{9}$$

The equation in (8) does not have any solution in $\mathbb{N}$ because, for every $k \in \mathbb{N}$, it is the case that $5^k + 1 > 2^{k+1}$. On the other hand, since $5^k > 2^{k+1} + 1$ for every $k \in \mathbb{N} \setminus \{1\}$, we obtain that the equation (9) admits exactly one solution in $\mathbb{N}$: $k = 1$.

Substituting back in (6) what we have just obtained for k, we conclude that $(n = 1, k = 1)$ is the only solution of that equation in $\mathbb{N} \times \mathbb{N}$; hence, $n = 1$ is the only natural number that is simultaneously a repunit and a triangular number. $\square$

We state below three questions one might easily pose after thinking for a while in possible variations of the preceding example; they all have to do with the determination of all the repunits in a *distinguished* sequence of natural numbers:

a) *Determine all the natural numbers n that belong to both the sequence of repunits and the sequence of pentagonal numbers (resp. hexagonal numbers, resp. heptagonal numbers, etc.).*

b) *Does there exist an odd perfect number in the sequence of repunits*?

c) *How many repunits are there in the Fibonacci sequence*?

The problem in item **a** is the variation of Example 5 which we obtain by replacing the sum $1+2+\cdots+n$ by $1+4+\cdots+(1+3(n-1))$ (resp. by $1+5+\cdots+(1+4(n-1))$, resp. by $1+6+\cdots+(1+5(n-1))$, etc.).

The question in item **b** is a *sui generis* one because, in point of fact, nobody knows whether or not odd perfect numbers exist; nevertheless, resorting to Euler's necessary condition for odd perfect numbers (cf. [2, Ch. 3, pp. 14-15]), it is not difficult to conclude that the answer for that question is in the negative.

Now then, what we know about the question in item **c** is that F. Luca proved in [5] that the only repunit in the Fibonacci sequence is $1 = F_1 = F_2$.

Finally, it may be worth mentioning that there are some variations of Example 5 that are wide-open yet: v.g., nobody knows how many repunits there are in the sequence of the prime numbers.

In the last problem that we are going to expound in our paper, the analysis of the discriminant of the auxiliary quadratic polynomial allows us to get started. In order to complete the solution, among other things, we are going to resort to the classical result on the solution set of the Pell equation (see, for instance, [1, pp. 121-124]).

Example 6. *(Turkish National Mathematical Olympiad; pb. 2, 2014 (2^{nd} round).) Find all positive integers $x, y,$ and z satisfying the equation $x^3 = 3^y \cdot 7^z + 8$.*

Solution. The equation can be rewritten as

$$(x-2)(x^2+2x+4) = 3^y \cdot 7^z.$$

Supposing that it has a solution in positive integers $x, y,$ and z, the fundamental theorem of arithmetic allows us to guarantee the existence of $a \in \mathbb{N}, b \in \mathbb{Z}^+$ such that

$$x^2 + 2x + (4 - 3^a \cdot 7^b) = 0.$$

This assumption also implies that the discriminant Δ of the quadratic polynomial $\mathcal{X}^2 + 2\mathcal{X} + (4 - 3^a \cdot 7^b)$ is a perfect square; since

$$\Delta = 4 - 4(4 - 3^a \cdot 7^b) = 4 \cdot 3^a \cdot 7^b - 12 = 4(3^a \cdot 7^b - 3) = 4(3)(3^{a-1} \cdot 7^b - 1),$$

we must have $a = 1$. Thus, the problem will be settled once we determine all those $b \in \mathbb{N}$ such that

$$7^b - 1 = 3s^2$$

for a certain $s \in \mathbb{N}$. In the light of the factorization $7^b - 1 = (7-1)(7^{b-1} + \cdots + 1)$, the equation in the previous line may be rewritten as

$$2(7^{b-1} + \cdots + 1) = s^2.$$

By considering the parity of the sum inside the parentheses we get that $b = 2\ell$ for some $\ell \in \mathbb{N}$. Hence, ℓ and s are natural numbers satisfying the equality $(7^\ell)^2 - 3s^2 = 1$; in other words, $(7^\ell, s)$ is a solution in positive integers to the Pell equation

$$X^2 - 3Y^2 = 1. \tag{10}$$

Resorting to the fact that the fundamental solution of (10) is $2+\sqrt{3}$, we will prove in the next paragraphs that, if $X_n + Y_n\sqrt{3}$ is a solution to (10) in which X_n is a power of 7, then $n = 2$. Clearly enough, this will yield that $b = 2$ and $x^2 + 2x + 4 = 3 \cdot 7^2$ which will allow us to conclude that $(x = 11, y = 3, z = 2)$ is the only triple satisfying the original equation.

Let us suppose that X_n and Y_n are natural numbers such that X_n is a power of 7 and $X_n + Y_n\sqrt{3}$ is a solution to (10).

If n is an odd natural number, then $n = 2k + 1$ for some $k \in \mathbb{Z}^+$; since

$$X_n + Y_n\sqrt{3} = \left(2+\sqrt{3}\right)^n = \left(2+\sqrt{3}\right)^{2k+1} = \sum_{j=0}^{2k+1} \binom{2k+1}{j} 2^{(2k+1)-j} \left(\sqrt{3}\right)^j,$$

it follows that

$$X_n = \sum_{\substack{0\le j\le 2k+1\\ 2\,|\,j}} \binom{2k+1}{j} 2^{(2k+1)-j} \left(\sqrt{3}\right)^j.$$

Hence, if n is an odd natural number, we have that $X_n \equiv 2 \pmod 3$; therefore, X_n can't be a power of 7 in this case.

If $4 \mid n$, then $n = 4k$ for some $k \in \mathbb{N}$. Thus,

$$X_n + Y_n\sqrt{3} = \left(2+\sqrt{3}\right)^n = \left(7+4\sqrt{3}\right)^{2k}$$

which implies that

$$X_n = \sum_{\substack{0\le j\le 2k\\ 2\,|\,j}} \binom{2k}{j} 7^{2k-j} \left(4\sqrt{3}\right)^j = \sum_{J=0}^{k} \binom{2k}{2J} 7^{2k-2J} \left(4\sqrt{3}\right)^{2J} = (48)^k + \sum_{J=0}^{k-1} \binom{2k}{2J} 7^{2(k-J)} \left(4\sqrt{3}\right)^{2J} \equiv (-1)^k \pmod 7.$$

Consequently, X_n can't be a power of 7 when n is a multiple of 4 either.

If $n = 4k + 2$ for some $k \in \mathbb{N}$, then

$$X_n + Y_n\sqrt{3} = \left(2+\sqrt{3}\right)^n = \left(7+4\sqrt{3}\right)^{2k+1}$$

and, consequently,

$$X_n = \sum_{\substack{0\le j\le 2k+1\\ 2\,|\,j}} \binom{2k+1}{j} 7^{(2k+1)-j} \left(4\sqrt{3}\right)^j = \sum_{J=0}^{k} \binom{2k+1}{2J} 7^{2k+1-2J} (48)^J$$

Let us suppose that $7^\alpha \parallel (2k+1)$ for some $\alpha \in \mathbb{N}$. We claim that $7^{\alpha+2}$ divides the first k terms of the sum

$$\sum_{J=0}^{k} \binom{2k+1}{2J} 7^{2(k-J)+1} (48)^J;$$

from this assertion and the fact that $7^{\alpha+1} \parallel 7\binom{2k+1}{2k}$, it will follow that X_n is a power of 7 only when $n = 2$.

Let $J \in \{0, 1, 2, \ldots, k-1\}$ and let β be the integer satisfying

$$7^\beta \le 2k + 1 - 2J < 7^{\beta+1}.$$

If $\alpha > \beta$, the exponent of 7 in the prime decomposition of $\binom{2k+1}{2J} = \frac{(2k+1)!}{(2k+1-2J)!(2J)!}$ is

$$\begin{aligned}
\sum_{\ell\in\mathbb{N}} \left(\left\lfloor \frac{2k+1}{7^\ell} \right\rfloor - \left\lfloor \frac{2J}{7^\ell} \right\rfloor - \left\lfloor \frac{2k+1-2J}{7^\ell} \right\rfloor \right) &\ge \sum_{\ell=\beta+1}^{\alpha} \left(\left\lfloor \frac{2k+1}{7^\ell} \right\rfloor - \left\lfloor \frac{2J}{7^\ell} \right\rfloor - \left\lfloor \frac{2k+1-2J}{7^\ell} \right\rfloor \right) \\
&= \sum_{\ell=\beta+1}^{\alpha} \left(\left\lfloor \frac{2k+1}{7^\ell} \right\rfloor - \left\lfloor \frac{2J}{7^\ell} \right\rfloor \right) \\
&= \alpha - \beta. \qquad (11)
\end{aligned}$$

It is clear that $7^{\alpha+2} \mid 7^{2k+1-2J}$ whenever that $2k+1-2J > \alpha+1$. Now then, if $2k+1-2J \leq \alpha+1$, we have that

$$\alpha \geq 2k - 2J \geq 7^{\beta} - 1 \geq \beta.$$

Since $\alpha \geq 1$, the previous chain of inequalities implies that $\alpha > \beta$; from this inequality and the bound in (11) we get that the exponent of 7 in the prime decomposition of $\binom{2k+1}{2J}7^{2k+1-2J}$ is greater than or equal to $(2k+1-2J)+(\alpha-\beta) \geq \alpha+2$. In both cases we obtain that $7^{\alpha+2} \mid \binom{2k+1}{2J}7^{2k+1-2J}(48)^J$ and whence our claim follows. □

4 Some problems for further thought

We provide now a list of problems that can be settled by resorting to the ideas and results discussed in this paper. We encourage the reader to give them all a try!

1. a) Let x, y be natural numbers such that $3x^2 + x = 4y^2 + y$. Prove that $x - y$ is a perfect square.

b) Let $x, y \in \mathbb{Z}$. If $x - y = x^2c - y^2d$ for some consecutive integers c and d, prove that $x - y$ is a perfect square.

2. Determine all pairs $(u, v) \in \mathbb{N} \times \mathbb{N}$ that satisfy the equation

$$u^3 = u^2 + 3uv + v^2.$$

3. ([6, p. 24]) Find all pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ for which the equality

$$\frac{1}{m} + \frac{1}{n} - \frac{1}{mn^2} = \frac{3}{4}$$

holds.

4. (M. Z. Garaev) Prove that the equation $x^3 + y^3 = (3xyz + 1)^2$ has no solution in natural numbers x, y, and z.

5. State and prove a theorem that encompasses both Example 2 and Exercise 1.

6. Solve any of the variations of Example 5 suggested in item a on page 7.

7. (*Mathematics Magazine*, **88** (June 2015), no. 3, p. 235.) Find all pairs of integers (x, y) such that

$$x^8 + (y^2 + y - 1)(4 - 3x^4) = 2.$$

8. (*USAMO; pb. 3, 1986.*) What is the smallest integer n, greater than one, for which the quadratic mean of the first n positive integers is an integer?

9. (*Tzaloa*, (2020), no. 1, p. 25.) For a natural number $m > 1$, let us denote with $p(m)$ the least prime factor of m. Suppose that a and b are integers greater than 1 satisfying $a^2 + b = p(a) + (p(b))^2$. Prove that $a = b$.

10. Find all solutions of $x^2 + xy + y^2 = x^2y^2$ in integers x, y.

11. ([4, p. 4]) Let r and s be non-zero integers. Prove that the equation

$$(r^2 - s^2)x^2 - 4rsxy - (r^2 - s^2)y^2 = 1$$

has no solutions in integers x and y.

12. Find all triples $(x, y, z) \in \mathbb{Z}^3$ such that $x^4 - 2z^2y^2 = z^4$.

References

[1] Titu Andreescu, Dorin Andrica, and Ion Cucurezeanu, *An introduction to Diophantine equations: a problem-based approach.* Birkhäuser Verlag, 2010.

[2] Leonhard Euler, *Tractatus de numerorum doctrina capita sedecim, quae supersunt.* Freely available online at: `http://eulerarchive.maa.org/pages/E792.html`

[3] John H. Jaroma, *Triangular repunit—there is but one.* Czechoslovak Math. J., **60** (2010), no. 4, pp. 1075-1077.

[4] Kenneth S. Williams and Kenneth Hardy, *The red book of mathematical problems.* Dover Publications Inc., Mineola, NY, USA, 1996.

[5] Florian Luca, *Fibonacci and Lucas numbers with only one distinct digit.* Port. Math., **57** (2000), no. 2, pp. 243-254.

[6] Anne Alberro Semerena, Radmila Bulajich Manfrino, and Carlos J. Rubio Barrios, *Problemas avanzados.* Serie: Cuadernos de Olimpiadas de Matemáticas. México D. F., 2010.

[7] J. Michael Steele, *The Cauchy-Schwarz master class.* Cambridge University Press, NY, USA, 2004.

[8] *Tzaloa: Revista de la Olimpiada Mexicana de Matemáticas*, (2016), no. 1. Freely available online at: `http://www.ommenlinea.org/wp-content/uploads/2017/04/Tzaloa_1_2016.pdf`

Acknowledgments. The author thanks Enrique Treviño for all the typos that he caught in a previous version of this document and for the suggestion that he provided which helped to finish the solution of the sixth example in a swifter manner. He also wishes to express his gratitude to Henry J. Ricardo for the remarks that he sent and which led to a cleaner rendition of the paper.

Author's affiliation: Escuela Superior de Matemáticas 2 - UAGro (Guerrero, México).
Email address: `josehernandez@uagro.mx`